

## Assignment 2, due September 26

**Corrections:** (none yet)

1. Suppose that  $\mathcal{F}$  and  $\mathcal{G}$  are  $\sigma$ -algebras and that all the information in  $\mathcal{G}$  also is in  $\mathcal{F}$ . Suppose that  $\Omega$  is a finite or countable (i.e. discrete) probability space with a probability function  $P(\omega)$  for  $\omega \in \Omega$ . Let  $X(\omega)$  be a real valued random variable. The conditional expectations, in the modern sense as random variables, are  $Y = E[X | \mathcal{F}]$  and  $Z = E[X | \mathcal{G}]$ . In each case, state whether the statement is true or false and explain your answer with a simple proof (explanation) or a counterexample.

- (a)  $Z \in \mathcal{F}$ .
- (b)  $Y \in \mathcal{G}$ .
- (c)  $Z = E[Y | \mathcal{G}]$ .
- (d)  $Y = E[Z | \mathcal{F}]$ .

2. Let  $\Omega$  be a discrete probability space and  $\mathcal{F}$  a  $\sigma$ -algebra. Let  $X(\omega)$  be a (function of a) random variable with  $E[X^2] < \infty$ . Let  $Y = E[X | \mathcal{F}]$ . The variance of  $X$  is  $\text{var}(X) = E[(X - \bar{X})^2]$ , where  $\bar{X} = E[X]$ .

- (a) Show directly from the (modern) definition of conditional expectation that

$$E[X^2] = E[(X - Y)^2] + E[Y^2]. \quad (1)$$

Note that this equation also could be written

$$E[X^2] = E \left[ (X - E[X | \mathcal{F}])^2 \right] + E \left[ (E[X | \mathcal{F}])^2 \right].$$

- (b) Use this to show that  $\text{var}(X) = \text{var}(X - Y) + \text{var}(Y)$ .
- (c) If we interpret conditional expectation as an orthogonal projection in a vector space, what theorem about orthogonality does part (a) represent?
- (d) We have  $n$  independent coin tosses with each equally likely to be H or T. Take  $X$  to be the indicator function of the event that the first toss is H. Take  $\mathcal{F}$  to be the algebra generated by the number of H tosses in all. Calculate each of the three quantities in (1) from scratch and check that the equation holds. Both of the terms on the right are easiest to do using the law of total probability, which is pretty obvious in this case.

3. Suppose the random variables  $Z_k$  are independent standard normals (Standard normal means  $Z_k \sim \mathcal{N}(0, 1)$ . Here  $\mathcal{N}(\mu, \sigma^2)$  means normal with mean  $\mu$  and variance  $\sigma^2$ .) Define a Gaussian random walk by  $X_k = Z_1 + \dots + Z_k$ , which is the same as  $X_0 = 0$  and  $X_{k+1} = X_k + Z_{k+1}$ . Use the path notation from class:  $X_{[1:t]} = (X_1, \dots, X_t)$ . Suppose in the following that  $1 \leq t \leq T$  and where appropriate that  $t + 1 \leq T$ .
- (a) Write a formula for the conditional probability density of  $X_{t+1}$  conditional on  $X_{[1:t]}$ . This takes the form of a formula for  $u(x_{t+1} | x_{[1:t]})$ . (Hint: conditional on  $x_{[1:t]}$ ,  $X_{t+1}$  is Gaussian with a certain mean and variance.) Which of the  $x_s$  for  $s \leq t$  does  $u(x_{t+1} | x_{[1:t]})$  actually depend on?
- (b) Use Bayes' rule and the formula from part (a) to write the joint density  $u(x_{[1:T]})$ . This is done by multiplying the conditional densities. We will use this formula many times during the semester but not again in this homework.
- (c) In this situation the random outcome is the path:  $\omega = X_{[0:T]}$ . Let  $V(\omega) = X_T^4$ . Calculate  $E[V]$ . Hint:  $\int_{-\infty}^{\infty} z^4 e^{-z^2/2} dz = -\int_{-\infty}^{\infty} z^3 \partial_z e^{-z^2/2} dz = 3 \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$ . Thus  $E[Z^4] = 3$  if  $Z \sim \mathcal{N}(0, 1)$ . More generally if  $Y \sim \mathcal{N}(\mu, \sigma^2)$  then we can represent  $Y$  as  $Y = \mu + \sigma Z$  and calculate  $E[Y^4]$ .
- (d) Let  $\mathcal{F}_t$  be the algebra generated by  $X_{[1:T]}$ . Then  $E[V | \mathcal{F}_t]$  is a function of  $X_{[1:T]}$ . Show that this actually is a function of just  $X_t$ , and not  $X_s$  for  $s < t$ . Calculate the function  $f(x, t)$  so that

$$f(X_t, t) = E[V | \mathcal{F}_t] .$$

- (e) Verify by the tower property formula for the conditional expectations  $E[V | \mathcal{F}_{t+1}]$  and  $E[V | \mathcal{F}_t]$ . Explain why this is the same as showing that

$$E[f(X_{t+1}, t+1) | \mathcal{F}_t] = f(X_t, t) .$$

Do this last calculation explicitly using the formula for  $f$ .