Assignment 3, due October 10

Corrections: (none yet)

1. An important part of the central limit theorem is that the distribution of $S_n = Y_n + \cdots + Y_n$ depends (to leading order in n) only on the mean and variance of Y (As usual, the Y_k are independent and have the same distribution as Y.). This exercise verifies this phenomenon for the fourth moment. Let $X_n = n^{-1/2}(S_n - n\mu)$ be the scaled deviation of S_n from its mean (Here $\mu = E[Y]$.). Define $\sigma^2 = \text{var}(Y) = E\left[(Y-\mu)^2\right]$, and $m_4 = E\left[(Y-\mu)^4\right]$. Assume $m_4 < \infty$ and show that

$$E\left[X_n^4\right] = 3\sigma^4 + O\left(\frac{1}{n}\right) \quad , \qquad \text{as } n \to \infty. \tag{1}$$

Hint: you can assume without loss of generality that $\mu = 0$ (why). Doing so gives

$$X_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \ .$$

so

$$E\left[X_n^2\right] = \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E\left[Y_j Y_k\right] \ .$$

Only the *diagonal* expectations, the ones with $j = k$ are different from zero. With the fourth moment, $E\left[Y_i Y_j Y_k Y_l\right] \neq 0$ only for a small subset of the possible values of i, j, k, l. Most of those expectations are σ^4.

For the non-math majors, $a_n = O(1/n)$ means that there is a number C so that $|a_n| \leq C/n$ for all n. It is the same to say that there is a C and an N_0 so that $|a_n| \leq C/n$ whenever $n > N_0$ (non-math majors, think this through.). These *big O* error bounds are useful because they are easy to manipulate and verify. For example, if $a_n = O(1/n)$ and $b_n = O(1/n)$, then $a_n + b_n = O(1/n)$, just take the $C_{a+b} = C_a + C_b$. It is easy to verify facts like $\left(1+n^2\right)^{-1/2} = O(1/n)$. If it looks proportional to *Const*$/n$ as $n \to \infty$, then it is $O(1/n)$.

2. The probability distribution of a Gaussian random variable is determined by its mean and variance. In particular, all higher moments are determined by the mean and variance.

(a) Show that if X is a one dimensional Gaussian with $E[X] = 0$ and $E[X^2] = \sigma^2$, then $E[X^4] = 3\sigma^4$. Hint: First write $X = \sigma Z$ where $Z \sim \mathcal{N}(0,1)$. Then show that the question reduces to showing $E[Z^4] = 3$ (This is an example of *non-dimensionalization.*). Finally, integrate by parts in the z integral, noting that $z^4 = z^3 z$ and $ze^{-z^2/2} = -\partial_z e^{-z^2/2}$.

(b) Assume that $X \sim \mathcal{N}(0, \sigma^2)$ and find a formula for $E[X^n]$ in terms of σ. The formula is different for even and odd n.