Assignment 4, due October 17

Corrections: (none yet)

1. A *standard* Brownian motion is a continuous[1] stochastic process X_t with $X_0 = 0$ and the conditional distribution of X_{t+s}, conditional on $\mathcal{F}_t$ given by $\Delta X = X_{t+s} - X_t \sim \mathcal{N}(0, s)$. The integral

$$I_f \;=\; \int_0^T f(t) dX_t$$

is defined as the limit

$$I_f \;=\; \lim_{\Delta t \to 0} \sum_{t_k < T} f(t_k) \Delta X_k \;, \tag{1}$$

where we use the usual notations $t_k = k\Delta t$, and $\Delta X_k = X_{t_{k+1}} = X_{t_k}$.

(a) Show that if the limit defining I_f exists and f is a given fixed (not random) continuous function of t, then I_f is Gaussian with mean zero and variance

$$E\big[\, I_f^2 \,\big] \;=\; \int_0^T f(t)^2 \, dt \;.$$

(b) Give a simple explicit check of this formula for the case $f(t) \equiv 1$. Write a formula for I_f in terms of X in that case.

(c) Generalize part (a) to get a formula for $E\left[I_f I_g \right]$.

(d) Show (assuming the limit exists and f' is continuous) that

$$I_f \;=\; f(T)X(T) \;-\; \int_0^T X_t f'(t) \, dt \;.$$

The integral on the right is a Riemann integral (the kind of integral you do in ordinary calculus). For this exercise you may use the approximation $\Delta f_k = f(t_{k+1}) - f(t_k) = f'(t_k)\Delta t$.

(e) Define

$$A_n \;=\; \int_0^T \sin\left(\frac{(n + \frac{1}{2})\pi t}{T} \right) \, X(t) \, dt \;.$$

Show that these are independent Gaussians and calculate their variances.

[1] There are several ways to state the hypothesis that a stochastic process is continuous. For now, just assume that the sample space Ω contains only continuous functions.

(f) (*Extra credit, only for those who know enough about Fourier series*) Show that the formula

$$X(t) \;=\; \sum_{n=0}^{\infty} c_n Z_n \sin\left(\frac{(n+\frac{1}{2})\pi t}{T}\right) \;,$$

with $Z_n \sim \mathcal{N}(0,1)$ independent and c_n constants related to part (e) defines a Brownian motion path up to time T.

2. In ordinary calculus we treat the squares of differentials as zero most of the time, but not in stochastic calculus.

(a) Suppose f and g are differentiable functions of t with bounded derivatives and define (using notation as above)

$$\int_0^T g(t)\left(df(t)^2\right) \;=\; \lim_{\Delta t \to 0} \sum_{t_k < T} g(t_k)\left(\Delta f_k\right)^2 \;.$$

Assuming $\Delta f_k \;=\; f'(t_k)\Delta t$, show that the limit is equal to zero. Informally, this is expressed by saying $(df)^2 = 0$.

(b) Let X_t be Brownian motion as above. Show that

$$\int_0^T \left(dX_t\right)^2 \;=\; \lim_{\Delta t \to 0} \sum_{t_k < T} \left(\Delta X_k\right)^2 \;=\; T \;.$$

(Hint: scale the ΔX_k^2 to be i.i.d. and use the law of large numbers.)

(c) Generalize this to

$$\int_0^T g(t)\left(dX_t\right)^2 \;=\; \lim_{\Delta t \to 0} \sum_{t_k < T} g(t_k)\left(\Delta X_k\right)^2 \;=\; \int_0^T g(t)\,dt \;.$$

It is more technical to do this completely because the random variables in the middle are not identically distributed. Instead, just show that the expected value converges to the integral and the variance converges to zero. (For those with the right probability background, this implies that the limit exists *in probability*.) Informally, this is expressed by saying that $(dX)^2 = dt$.

3. The cumulative normal is (as usual, $Z \sim \mathcal{N}(0,1)$)

$$N(x) \;=\; \Pr\left(Z < x\right) \;=\; \int_{-\infty}^{x} e^{-z^2/2}\,\frac{dz}{\sqrt{2\pi}} \;.$$

This satisfies $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. It has the obvious properties $N(x) \to 0$ as $x \to -\infty$ and $N(x) \to 1$ as $x \to \infty$. Define a random variable

$$H_t \;=\; \Pr(\, X_T < 0 \mid \mathcal{F}_t) \;.$$

Here $\mathcal{F}_t$ is the $\sigma-$algebra generated by the values of X_s for $0 \le s \le t$. Assume that $0 < t \le T$.

(a) Show that

$$H_t \;=\; f(X_t, t)\;,$$

where

$$f(x,t) \;=\; N\left(\frac{x}{\sqrt{T-t}}\right)\;. \tag{2}$$

Hint: The increment $X_T - X_t$ is independent of all the information in $\mathcal{F}_t$ by the independent increments property. There is a formula for $\Pr(Y < x)$ in terms of the cumulative normal when $Y \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Show that this function $f(x,t)$ satisfies the partial differential equation

$$\partial_t f \;+\; \frac{1}{2}\partial_x^2 f \;=\; 0\;.$$

Do this by explicitly calculating the t and x derivatives of the formula (2).