Stochastic Calculus, Courant Institute, Fall 2011
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2011/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 5, due October 24

Corrections: (none yet)

See notes on the Ito Integral posted on the Resources page.

1. Find a formula for $\int_0^t \left(W_s^2 - s\right) dW_s$ in terms of W_t. Hint: use Ito's lemma to calculate the differentials of W_t^3 and tW_t.

2. The stochastic process X_t is *geometric Brownian motion* if it satisfies a stochastic differential equation (see the notes) of the form $dX_t = \mu X_t dt + \sigma X_t dW_t$. Show that stochastic processes of the form $X_t = X_0 e^{\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t}$ are geometric Brownian motions.

3. Suppose $X_t = \int_0^t F_s dW_s + \int_0^t G_s ds$.

   (a) Show that the Ito integral is additive in the sense that $\int_0^{t'} F_s dW_s = \int_0^t F_s dW_s + \int_t^{t'} F_s dW_s$. For this you need to define approximations that do not start at $t = 0$ and show that additivity holds approximately for the approximations.

   (b) If $t' = t + \Delta t$ for small Δt, you may use the approximations $F_s \approx F_t$ and $G_s \approx G_t$ for $t \leq s \leq t'$. The increment of X over this interval is $\Delta X = X_{t+\Delta t} - X_t$. Show that

$$\begin{aligned} E\left[\Delta X \mid \mathcal{F}_t\right] &\approx G_t \Delta t \\ E\left[(\Delta X)^2 \mid \mathcal{F}_t\right] &\approx F_t^2 \Delta t \\ \operatorname{var}\left[\Delta X \mid \mathcal{F}_t\right] &\approx F_t^2 \Delta t \end{aligned}$$

   The conditional variance in the last formula is the conditional expected value of the square difference from the conditional mean. These formulas may be written informally as $E[dX_t \mid \mathcal{F}_t] = G_t dt$ and $E[(dX_t)^2 \mid \mathcal{F}_t] = F_t^2 dt$.

   (c) (*not an action item*) The formulas of part (b) usually are used in reverse. In modeling we have estimates of $E[\Delta X \mid \mathcal{F}_t]$ and $E[\Delta X^2 \mid \mathcal{F}_t]$ and use them to discover F and G, or the coefficients $a(X)$ and $b(X)$ in the stochastic differential equation modeling X. For example, the geometric Brownian motion of a stock price is "derived" ("motivated might be more accurate) by saying $E[(\Delta X/X_t) \mid \mathcal{F}_t] = \mu \Delta t$ (rate of return is $\Delta X/(X_t \Delta t)$. This is constant expected rate of return $= \mu$.) Similarly, $\operatorname{var}(\Delta X/X_t \mid \mathcal{F}_t) = \sigma^2 \Delta t$ defines σ as the *volatility*.

4. The *Ornstein Uhlenbeck* process is the linear stochastic differential equation

$$X_t \;=\; -\gamma X_t\, dt \,+\, \sigma dW_t \;. \tag{1}$$

(a) Show that the solution is given by a formula something like (part of the problem is to find the correct formula)

$$X_t \;=\; e^{-\gamma t} X_0 \,+\, \int_0^t e^{-\gamma(t-s)} dW_s \;. \tag{2}$$

(b) Assuming that X_0 is not random, use the (corrected) formula (2) to find formulas for $m_t = E[X_t]$ and $s_t^2 = E[X_t^2]$. Evaluate these in the limit $t \to \infty$.

(c) As an alternative method, calculate dm_t by formally differentiating $m = E[X]$, as $dm_t = E[\,E[\,dX_t \mid \mathcal{F}_t]\,]$. Show that you get the same result by using the first formula of part (3b) and taking the limit $\Delta t \to 0$.

(d) Find formula for ds_t in a similar way using the middle part of (3b). For this, write $(X_t + \Delta X)^2 \,=\, X_t^2 \,+\, 2X_t\Delta X \,+\, \Delta X^2$ and take the conditional expectation in $\mathcal{F}_t$.

(e) Intuition might suggest that the probability distribution of X_t would converge so a limiting distribution as $t \to \infty$. The term $-\gamma X_t dt$ acts as a restoring force, pushing X_t toward zero from the positive and negative directions. The restoring force becomes stronger as $|X_t|$ increases but the noise keeps the same strength. If X_t were in equilibrium, we would have $dm_t = 0$ and $ds_t = 0$. Use the results of part (d) to find the values of m_t and s_t corresponding to this steady state. If everything is correct, these values should agree with those of part (b).