Stochastic Calculus, Courant Institute, Fall 2011

http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2011/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 6, due November 7

Corrections: (none yet)

See notes on the Ito Integral and diffusions posted on the Resources page.

1. Consider the incorrect approximation formula for $\int_0^t W_s dW_s$:

$$Y_t^{\Delta t} \;=\; \sum_{t_k<t} W_{t_{k+1}} \left(W_{t_{k+1}} - W_{t_k} \right) \;. \tag{1}$$

   Repeat (changing as necessary) the steps that lead to

$$X_t^{\Delta t} \;\to\; X_t \;=\; \frac{1}{2} \left(W_t^2 - t \right)$$

   to find a formula for

$$Y_t \;=\; \lim_{\Delta t \to 0} Y_t^{\Delta t} \;.$$

   Show that $Y_t \neq X_t$. Explain the difference by calculating

$$E\left[W_{t_{k+1}} \left(W_{t_{k+1}} - W_{t_k} \right) \right] \;\neq\; E\left[W_{t_k} \left(W_{t_{k+1}} - W_{t_k} \right) \right] \;.$$

   This calculation predicts a formula for $Y_t - X_t$. Find it.

2. The *urn process* (or one of the processes called urn process) has a number of red and blue balls totaling n. The number of red balls at time t_k is N_k. At each time, you remove one of the balls (chosen independently and at random, with each ball equally likely to be chosen) and replace it with a new independent ball with probability p to be blue and $1-p$ to be red.

   (a) Calculate the probabilities for $N \to N+1$ and $N \to N-1$ (and therefore $N \to N$) in one step of the process. Use this to calculate $E[\,\Delta N \mid \mathcal{F}_k]$ and $E\big[\,(\Delta N)^2 \mid \mathcal{F}_k\big]$.

   (b) Write an SDE for N_t assuming $\Delta t = 1$ and $t_k = k\Delta t$. Take $dt = 1$ in this calculation. This calculation is sort of correct and sort of incorrect. We will refine it by using systematic *scalings* next week.

3. Consider a chemical system in which there are N_t *copies* of a certain kind of molecule at time t. In a time dt, there is probability λdt that a new molecule of this type "arrives" (is produced by some chemical reaction). Molecules of this type are removed by combining with each other to form

dimers (compound molecules consisting of two of the originals bound together). The probability that a given molecule can find a partner and form a dimer is $\mu(N_t - 1)dt$. Here $N_t - 1$ i the number of other molecules that this one could combine with. Write an approximate SDE for this process, assuming N_t is large. The difference between N_t and $N_t - 1$ should not matter if N is large.

4. The *Ornstein Uhlenbeck* process is the one we saw a few weeks ago: $dV_t = -\gamma V_t dt + \sigma dW_t$. Suppose $X_0 = 0$ and $dX_t = V_t dt$. Show that for large t, $E\left[X_t^2\right] \approx Dt$ and find a formula for D in terms of γ and σ. Hint: express X_t as a double integral of dW_t and reverse the order of integration. Use the previous expression for V_t as the first step. This formula was first derived by Einstein in his model of Brownian motion.