Stochastic Calculus, Courant Institute, Fall 2011
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2011/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 7, due December 5

Corrections: (none yet)

1. Show that the discussion of the existence and value of quadratic variation holds even when $G)t \neq 0$. The notes claim this but give no details.
   (a) Show (somewhat informally in the style of the notes) that we still have
   $$\sum_{k,j<k} E[R_j R_k] \;=\; O(\Delta t)\;.$$
   Although $E\left[R_k \mid \mathcal{F}_{t_k}\right] \neq 0$, you can say approximately what it's value is and use that to evaluate (approximately) the sum over k above. In the overall argument, use the fact that
   $$E[R_j R_k \mid \mathcal{F}_{t_k}] \;=\; R_j E[R_k \mid \mathcal{F}_{t_k}]\;,$$
   if $j < k$.
   (b) Figure out the power of Δt in the best bound for
   $$E\left[\left\{\left(X_{t_{k+1}} - X_{t_k}\right)^2 - F_{t_k}^2 \Delta t\right\}^2\right]\;.$$
   Is it different when $G \neq 0$. Does it invalidate the discussion in the notes?

2. Suppose S_t is a geometric Brownian motion, which means that
   $$dS_t \;=\; \mu S_t dt \,+\, \sigma S_t dW_t\;,$$
   for some constants μ and σ.
   (a) Let the expected value of S_t be M_t. Show that $M_t = e^{\mu t} M_0$ by expressing $dE[M_t]$ in terms of M_t. Of course it is possible to do this using the well known formula $S_t = S_0 e^{\cdots}$, but please don't.
   (b) Let N_t be $N_t = E[S_t^2]$. Find a formula for N_t in terms of N_0 by calculating $dE[S^2]$ using the Ito lemma for general diffusions. Please don't use $S_t = S_0 e^{\cdots}$.
   (c) Find a formula for
   $$Y_T \;=\; \int_0^T S_t dS_t$$
   in terms of S_T and a Riemann integral. You can get the formula for Y_T by seeing what it takes to get $dY_t = S_t dS_t$.

(d) You can get a formula for $E[Y_T]$ by integrating $E[S_t dS_t]$ and using part (a). You can get a different formula using the answers to parts (b) and (c). Show that these agree.

3. Suppose S_t is a geometric Brownian motion and $X_t = u(S_t, t)$ that satisfies

$$\frac{dX_t}{X_t} = 2\,\frac{dS_t}{S_t} \; .$$

What could the function $u(s,t)$ look like?

4. This problem addresses a frequent misunderstanding that even a math professor may have been subject to at one time.

(a) Suppose that X_t and Y_t are two diffusions defined on the same "space" with the same filtration. Show that $X_t + Y_t$ is a diffusion. (This is very easy.)

(b) Suppose that X_t and Y_t are independent geometric Brownian motions with $\mu_X \neq \mu_Y$ and $\sigma_X \neq \sigma_Y$. Is $Z_t = X_t + Y_t$ a Markov process?