Stochastic Calculus, Courant Institute, Fall 2011
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2011/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 8, due December 14

Corrections: (none yet)

1. Suppose X_t is a one dimensional diffusion that satisfies the SDE $dX_t = adt + bdW_t$ where a and b are fixed constants. Suppose that $X_0 = 0$.

   (a) Give a formula for X_t in terms of W_t and use that to describe the distribution of X_t (what kind of distribution, what mean and variance).

   (b) Use the answer to part (a) to write a formula for $u(x,t)$, the probability density of X_t.

   (c) Write the forward equation for this SDE and verify by explicit calculation that this $u(x,t)$ is a solution.

2. Suppose X_t is an n dimensional diffusion that satisfies $dX_t = bdW_t$ where b is a fixed constant $n \times n$ non-singular matrix. Suppose that $X_0 = 0$.

   (a) Give a formula for X_t in terms of W_t and b. Recall that both X_t and W_t are n component vectors.

   (b) Use the formula of part (a) to describe the distribution of X_t – its mean and covariance and type of distribution.

   (c) Use the result of part (c) to give a formula for $u(x,t)$, the probability density of X_t. For this formula, note that if Y is Gaussian with mean zero and covariance C, then the probability density of Y is

   $$u(y) = \frac{1}{(2\pi)^{n/2}\det(C)^{1/2}} e^{-y^t Hy/2} .$$

   where $H = C^{-1}$.

   (d) Show that the formula from part (c) satisfies the forward equation. Here are some hints for the algebra.

   $$\partial_{y_i} u(y) = -u(y)\partial_{y_i}\left(y^t Hy/2\right) ,$$

   and

   $$\partial_{y_j}\partial_{y_i} u = u(y)\left[\partial_{y_j}\left(y^t Hy/2\right)\right]\left[\partial_{y_i}\left(y^t Hy/2\right)\right] - u(y)\partial_{y_j}\partial_{y_i}\left(y^t Hy/2\right) .$$

But

$$y^t H y \;=\; H_{ii} y_i^2 \,+\, 2 \sum_{k=1, k\neq i}^{k=n} H_{ik} y_i y_k \;+\; \text{stuff not involving } y_i ,$$

so

$$\partial_{y_i} y^t H y \;=\; 2 H_{ii} y_i \,+\, 2 \sum_{k=1, k\neq i}^{k=n} H_{ik} y_k \;=\; 2 \sum_{k=1}^{n} H_{ik} y_k \;=\; 2\left(Hy\right)_i \;,$$

and

$$\partial_{y_j} \partial_{y_i} y^t H y \;=\; 2 H_{ij} \;.$$

((*not to hand in*) This formula is obviously false if H is not symmetric. What is the correct formula for non-symmetric H? Where in the derivation was symmetry used?) In components,

$$\left[\partial_{y_j}\left(y^t H y/2\right)\right]\left[\partial_{y_i}\left(y^t H y/2\right)\right] \;=\; \sum_{k,l} H_{ik} H_{jl} y_k y_l \;.$$

Also, the formula $HC = I$ in components is (the order of subscripts does not matter when C and H are symmetric)

$$\sum_{k=1}^{n} C_{ik} H_{jk} \;=\; \delta_{ij} \;.$$

3. Consider the one dimensional *Ornstein Uhlenbeck* mean reverting process $dX_t = -\gamma X_t dt + b dW_t$. Suppose $X_0 = x_0$ is a fixed number, probably non-zero.

   (a) Write equations for the mean and variance of X_t. Call them $\mu(t)$ and $\sigma^2(t)$. Find the limits of $\mu(t)$ and $\sigma^2(t)$ as $t \to \infty$. Call them simply μ and σ^2.

   (b) Use the result of part (a), and the fact that X_t is Gaussian, to write a formula for $u(x,t)$, the probability density for X_t. Verify by explicit differentiation that it satisfies the forward equation for the Ornstein Uhlenbeck process.

   (c) A probability density function $u(x,t)$ is *time invariant* if it does not depend on t. Show that the Gaussian density with mean μ and variance σ^2 from part (a) (the limiting values) is a time invariant solution of the forward equation. This is called a *steady state* probability density. In finance, models that have steady state probability densities are called *equilibrium* models. The term is used even when X_t does not have the steady state density, because the time dependent density converges to the steady state density as $t \to \infty$.

4. This concerns the backward equation and duality in a specific example.

(a) Write the backward equation for the Ornstein Uhlenbeck process of problem (3).

(b) Suppose the time T payout is $V(x) = (x - a)^2$. Show that the backward equation of part (a) has a solution of the form $f(x,t) = B(t)(x - A(t))^2 + C(t)$. For this, you need only substitute the *ansatz* (assumed form of the solution) into the equation and find differential equations for $A(t)$, $B(t)$, and C that make the partial differential equation hold.

(c) Solve the ODE's of part (b) to get explicit formulas for $A(t)$, $B(t)$, and $C(t)$. Use the known values at $t = T$ as "initial conditions".

(d) What is the limit as $T \to \infty$ of $B(t)$? What does this tell you about how $E_{x,t}[V(X_T)]$ depends on T for large T? How is this consistent with the idea that the distribution of X_T is approximately the steady state distribution that does not depend on X_t?

(e) Suppose $E_{\mu,\sigma^2}[g(X)]$ refers to the expected value of $g(X)$ when $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for $E_{\mu,\sigma^2}[(X - a)^2]$. Hint: use $(x - a)^2 = \big((x - a) + (a - \mu)\big)^2$, calculate the three terms of the square, then take expectations.

(f) The duality used to derive the forward equation from the backward equation is that the expected value of $f(X_t, t)$ in the probability density $u(x,t)$ does not depend on t. Check that explicitly in this example. The density of X_t is normal with parameters known from problem (3a). The value function $f(x,t)$ is quadratic with parameters known from part (c). Use the formula of part (e) to show that the result is independent of t.

5. This exercise gives a different way to derive the backward equation that makes it easy to solve related problems.

   (a) Suppose M is any random variable,[1] $\mathcal{F}_t$ is a filtration, and $M_t = E[M \mid \mathcal{F}_t]$. Show that if $t' > t$, then $E\left[M_{t'} \mid \mathcal{F}_t\right] = M_t$. A stochastic process with this property is a *martingale*.

   (b) Suppose that M_t is a diffusion process that satisfies $dM_t = A_t dt + B_t dW_t$. Show that M_t is a martingale if $A_t = 0$. Hint: the class material does not suffice to do this completely rigorously. But you can use Ito arguments to calculate $E[M_{t+\Delta t} \mid \mathcal{F}_t]$ and argue from there.

   (c) Suppose X_t satisfies a one dimensional SDE $dX_t = a(X_t)dt + b(X_t)dW_t$, and that $g(x,t)$ is differentiable. Use the Ito calculus to find an expression of the form $dg(X_t,t) = A_t dt + B_t dW_t$. This requires two steps. First you use the Ito's lemma appropriate for general diffusions to express dg in terms of dX and dt. Then you use the SDE to convert this into an expression in terms of dW and dt.

[1] Well, not absolutely *any* random variable. Just ignore the possible fine print in this exercise.

(d) Let $M = V(X_T)$ be the random variable. Because X_t is a Markov process, you know that M_t depends only on X_t and not on earlier values X_s for $s < t$. Therefore, there is a function $f(x,t)$ so that $M_t = f(X_t, t)$. Use part (c) to find the partial differential equation that f must satisfy in order for $f(X_t, t)$ to be a martingale. If you do it right, this is the backward equation.

6. Here is an example of the use of the reasoning of problem (5). Let

$$M = \int_0^T V(X_s) ds \; .$$

Then $M_t = K_t + U_t$ is the sum of a known part in $\mathcal{F}_t$, which is

$$K_t = \int_0^t V(X_s) ds \; , \tag{1}$$

and the unknown part

$$U_t = f(X_t, t) \; ,$$

$$f(x,t) = E_{x,t}\left[\int_t^T V(X_s) ds \right] \; . \tag{2}$$

(a) Use the martingale property of M_t to calculate A_t in the expression $dU_t = A_t dt + B_t dW_t$. Hint: use the Ito differential of the Riemann integral (1).

(b) Use the result of part (a) and the calculations of problem (5c) to find the differential equation that $f(x,t)$ must satisfy in order for $K_t + U_t$ to be a martingale. This is a derivation of the PDE satisfied by the quantity (2).

7. (*not to hand in*) Enjoy the fact that there are no more exercises for this class this semester.