Sample questions for the final exam

Instructions for the final:

- The final is Monday, December 17 from 7:10 to 9pm
- Explain all answers, possibly briefly. A correct answer with no explanation may receive no credit.
- You will get 20% credit for not answering a question. Points will be subtracted if you give a wrong answer.
- Cross off anything you think is wrong. You will have points subtracted for wrong answers even if the correct answer also appears.
- You may use one $8\frac{1}{2} \times 11$ piece of paper, a *cheat sheet* with anything you want written on it.
- The actual exam will be shorter than this.

True/False. In each case, state whether the statement is true of false and give a few words or sentence of explanation.

1. Suppose $Z = (Z_1, \ldots, Z_n)^2$ with the $Z_k \sim \mathcal{N}(0,1)$ independent. Suppose that A and B are $n \times n$ non-singular matrices and that $A \neq B$. Suppose $X = AZ$ and $Y = BZ$. Then $X \neq Y$ almost surely.
2. Suppose $Z = (Z_1, \ldots, Z_n)^2$ with the $Z_k \sim \mathcal{N}(0,1)$ independent. Suppose that A and B are $n \times n$ non-singular matrices and that $A \neq B$. Suppose $X = AZ$ and $Y = BZ$. Then X and Y have different probability distributions.
3. Suppose $\mathcal{F}_t$ is a filtration on a probability space Ω. Suppose X is a random variable and $X_t = E[X \mid \mathcal{F}_t]$. Then X_t is a Markov chain.
4. If X_t is a martingale, then X_t is a Markov chain. (Give a counter-example if it's false.)
5. If $dX_t = \gamma_t X_t dt + dW_t$ and $dY_t = \lambda Y_t dt + dX_t$, then Y_t is an Ornstein Uhlenbeck process.
6. If $dX_t = \gamma_t X_t dt + dW_t$ and $dY_t = \lambda Y_t dt + dX_t$, then Y_t is a Markov process.
7. If X_k is a sequence of random variables and $\mathrm{P}(X_k \neq 0) \to 0$ as $k \to \infty$, then $X_k \to 0$ as $k \to \infty$.

8. If S_t is a geometric Brownian motion then S_t^3 is a geometric Brownian motion.

9. If $dX_t = X_t dt + \sigma dW_t$ and $dY_t = -Y_t dt + \sigma dW_t$, then $[X]_t$ and $[Y]_t$ (quadratic variations) are the same.

10. If (X_t, Y_t) is a two dimensional diffusion process, then X_t is a one dimensional diffusion.

11. If (X_t, Y_t) is a two dimensional diffusion, then X_t is a one dimensional stochastic process.

12. If X_t is a continuous time martingale and Y_t is equivalent to X_t (more precisely, the probability measure P for $X_{[0,T]}$ is equivalent to the probability measure Q for $Y_{[0,T]}$), then Y_t is a martingale.

13. If X_t is a continuous time Markov process and Y_t is equivalent to X_t (more precisely, the probability measure P for $X_{[0,T]}$ is equivalent to the probability measure Q for $Y_{[0,T]}$), then Y_t is a Markov process.

Multiple choice In each case only one of the possible answers is correct. Identify the correct answer and explain why.

14. Suppose X_t is a Markov process, and $t_1 < t_2 < T$. Suppose the $\sigma-$algebra $\mathcal{F}_t$ is generated by X_s for $s \leq t$ and the $\sigma-$algebra $\mathcal{G}_t$ is generated by X_t only. Which of the following is a consequence of the *tower property*

    (a) $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

    (b) There is a function $f(x,t)$ so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.

    (c) $f(x, t_1) = E_{x,t_1}[f(X_{t_2}, t_2)]$, where $f(x,t)$ is the function so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.

    (d) $E[V(X_T) \mid \mathcal{F}_t] = E[V(X_T) \mid \mathcal{G}_t]$

15. Suppose X_t and Y_t are geometric Brownian motions that satisfy $dX_t = \mu_X X_t dt + \sigma_X X_t dW_{1,t}$, and $dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_{2,t}$. Here $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions, $\mu_X \neq \mu_Y$ and $\sigma_X \neq \sigma_Y$. Let $S_t = X_t + Y_t$. Then

    (a) S_t is a Markov process but not a diffusion process.

    (b) S_t is a geometric Brownian motion.

    (c) S_t is a diffusion and a Markov process.

    (d) S_t is a diffusion process but not a Markov process.

16. Suppose $dX_t = X_t^2 dt + \sqrt{X_t} dW_t$. Which of the following processes produces a measure that is equivalent to this one?

    (a) $dX_t = X_t^2 dt + 2\sqrt{X_t} dW_t$.

(b) $dX_t = \sqrt{X_t}dW_t$.

(c) $dX_t = X_t^2dt + X_tdW_t$.

(d) $dX_t = X_t^2dt - \sqrt{X_t}dW_t$.

(e) (a) and (b)

(f) (b) and (d)

17. Which is true about L given below?

$$L = \begin{pmatrix} -.2 & .1 & .1 \\ .4 & -.7 & .3 \\ .4 & 0 & -.4 \end{pmatrix} .$$

(a) L could be the transition probability matrix of a discrete time Markov chain with a discrete state space.

(b) L could be the transition probabilitymatrix of a discrete time Markov chain with a continuous state space.

(c) L could be the transition rate matrix of a continuous time Markov chain with a discrete state space.

(d) L could be the transition rate matrix of a continuous time Markov chain with a continuous state space.

(e) L could be the generator of a diffusion process.

(f) None of the above.

General questions.

18. Suppose X_t satisfies the SDE $dX_t = adt + dW_t$ (a is a constant) with $X_0 = 0$. What is the distribution of X_3 given that $X_1 = -1$, $X_4 = 2$, and $X_5 = 1$? Give the type of distribution and the parameters that describe the conditional distribution of X_3 completely.

19. If X is an exponential random variable with $m = E[X]$, then $E[X^3] = Cm^3$, where C does not depend on m. Find a formula for C.

20. Suppose $X = \int_0^1 tdW_t$ and $Y = \int_0^1 t^2dW_t$. Calculate $E[X^2]$ and $\text{cov}(X,Y)$.

21. Suppose $dX_t = a(X_t)dt + b(X_t)dW_t$. Suppose $s = t^2$ and $Y_s = X_{t^2}$. Find the SDE that Y_s satisfies in the weak sense. Hint: calculate $E[dY \mid \mathcal{F}_s]$, where $dY = Y_{s+ds} - Y_s$, etc.

22. Suppose X_t is an Ornstein Uhlenbeck process $dX_t = -\gamma X_tdt + \sigma dW_t$. Calculate $d\cos(X_t)$ using the Ito calculus.

23. Give an example of a continuous time Markov process whose sample paths are not continuous functions of time.

24. Consider the limits (in our usual notations)

$$Y = \lim_{\Delta t \to 0} \sum_{t_k < T} W_{t_k}(W_{t_{k+1}} - W_{t_k})$$

$$Z = \lim_{\Delta t \to 0} \sum_{t_k < T} W_{t_{k+1}}(W_{t_{k+1}} - W_{t_k})$$

Either show that $Y = Z$ (almost surely) or find a formula for $Z - Y$.

25. Suppose $dX_t = \mu X_t dt + \sigma X_t dW_t$ and that $u(x,t)$ is the probability density of X_t. Write the partial differential equation that u satisfies.

26. A mean reverting geometric Brownian motion is defined by a system of the equations: $dX_t = \mu(X_t, \overline{X}_t)X_t dX_t + \sigma X_t dW_t$, and $d\overline{X}_t = \lambda(X_t - \overline{X}_t)dt$. Suppose we want to calculate $f(x, \overline{x}, t) = E_{x,\overline{x},t}[V(X_T)]$. Write the partial differential equation satisfied by f.

27. Define

$$f(x,t) = \mathrm{E}_{x,t}\left[\exp\left(\mu \int_t^T X_s ds\right)\right] .$$

Suppose $dX_t = -X_t dt + dW_t$. Find a formula for f. Do this in two ways:

(a) The exponent is Gaussian. Find its mean and variance.

(b) Write the backward equation (Feynman Kac), solve it by an exponential ansatz.

28. Suppose $u_0(x) = 2x$ for $0 \leq x \leq 1$ and $u_0(x) = 0$ otherwise. Suppose X_t is a Brownian motion with $X_0 \sim u_0$. Write the probability density for X_t.

29. Suppose X_t is standard Brownian motion with $X_0 = 0$. Let τ be the first time $X_t = r$. Here r is a positive parameter in the problem.

(a) What is $\mathrm{E}[\tau]$?

(b) Continue with part (a), find a formula for $f(x) = \mathrm{E}_x[\tau]$. Which is the expected value of τ starting from $X_0 = x$, of course with $x \leq r$.

(c) Formulate a PDE problem that characterizes the probability density of a particle X_t that has $\tau > t$. Suppose $X_{\tau \wedge t}$ is the stopped process and $u(x,t)$ is the probability density of $X_{\tau \wedge t}$ for $|x| < r$ (particles that have not yet been stopped). What PDE does u satisfy? What are its boundary conditions, initial conditions, final conditions, etc.?

30. Write a formula for the likelihood ratio $L(S_{0,T})$ that transforms margingale geometric Brownian motion $dS = \sigma S dW_t$ to a geometric Brownian motion with positive expected growth $dX = \mu S dt + dW$.