Stochastic Calculus, Courant Institute, Fall 2012
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2012/index.html
Always check the class message board on the blackboard site from home.nyu.edu before doing any work on the assignment.

Assignment 7, due December 10

Corrections: (none yet.)

1. Suppose we want to evaluate $A = \mathrm{E}\left[e^{-X_T^2/2}\right]$ where X_t is a standard Brownian motion starting from $X_0 = 0$. One approach is to simulate N Brownian motion paths and use the estimator

$$\widehat{A} = \frac{1}{N}\sum_{k=1}^{N} e^{-X_{k,T}^2/2} \; . \tag{1}$$

Another approach is to simulate the Ornstein Uhlenbeck process

$$dX = -\gamma X_t dt + dW_t \; .$$

Then there is a change of measure formula $L(X)$ so that

$$A = \mathrm{E}_{\mathrm{OU}}\left[e^{-X_{k,T}^2/2} L(X_{[0,T]})\right] \; . \tag{2}$$

Another way to estimate A is to simulate N Ornstein Uhlenbeck paths use

$$\widehat{A} = \frac{1}{N}\sum_{k=1}^{N} e^{-X_{k,T}^2/2} L(X_{k,[0,T]}) \; . \tag{3}$$

The second approach is more complicated, but it could be a better estimator for large T.

   (a) Write an analytic formula for A as a function of T.
   (b) Write a formula for the variance of the estimator (1).
   (c) Use these to show that the relative accuracy of the Monte Carlo estimator gets worse as T increases. Give an intuitive explanation for this in terms of the distribution of X_T and the range of values of X_T that contribute most to A. Make your explanation quantitative (giving the right power of T) if you can.
   (d) Write a formula for L in (2) that gives the correct A. This is an application of Girsanov's formula.
   (e) That formula involves

$$Y_t = \int_0^t X_t dX_t$$

   when X_t is the Ornstein Uhlenbeck process. Find an explicit expression for Y_t.

(f) Use your answer to part (e) to find an explicit formula for A in terms of the OU process. This should agree with your answer to part (a).

2. Suppose X_t is Brownian motion with $X_0 = 1$. Let τ be the stopping time that is the first time $X_t = 0$. On previous assignments we have studied hitting probabilities.

   (a) Write a formula for the probability density for X_t conditional on $\tau > t$.

   (b) Show by explicit calculation that

$$\mathrm{E}[\, X_t \mid \tau > X_t] = \mathrm{P}(\tau \leq t) \ .$$

   (c) Use the result of part (b) to show that the stopped process $X_{t\wedge\tau}$ satisfies $\mathrm{E}[\, X_{t\wedge\tau}] = X_0$.

3. Consider the stochastic differential equation

$$dX_t = -\gamma X_t dt + \sigma\sqrt{X_t} dW_t \ . \qquad (4)$$

   wit $X_0 = 1$.

   (a) Give a qualitative derivation of (4) by thinking of a large number of people waiting in a line. Let N_k be the number of people waiting in line at step k. Suppose N_k is a large number At time k, everyone in the line tosses a coin, all independent, and leaves with probability ϵ. Find a scaling of ϵ and t with N so that time dt corresponds to $k \to k+1$ and the scaled N_k converges in distribution to the process (4). This just means finding a scaling factor $r(\epsilon)$ and $s(\epsilon)$ (both powers of ϵ) so that $\mathrm{E}[\, dX_t]$ and $\mathrm{E}\left[\, dX_t^2\right]$ are both of order dt.

   (b) Write a program in `R` to simulate the process (4) up to time $t = 1$. Plot a histogram of the distribution of X_1 (take $\gamma = .5$ and $\sigma = 1$). Show that the histogram is incorrect if Δt is too large, but seems to have a limit as $\Delta t \to 0$.