Stochastic Calculus Assignment 1

Chen-Hung Wu

December 10, 2012

1. **(a)** If $X \sim \mathcal{N}(0, \sigma^2)$, then

$$\begin{aligned} E[X^4] &= \int_{\mathbb{R}} x^4 e^{-\frac{x^2}{2\sigma^2}}\,dx \\ &= \int_{\mathbb{R}} x^3(-\sigma^2)\partial_x e^{-\frac{x^2}{2\sigma^2}}\,dx \\ &= (-\sigma^2)x^3 e^{-\frac{x^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} + 3\sigma^2 \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2\sigma^2}}\,dx \\ &= 0 + 3\sigma^4 = 3\sigma^4. \end{aligned}$$

(b) Following from **1(a),** the variance of X^2 is

$$\begin{aligned} \operatorname{var}\left(X^2\right) &= E\left[X^4\right] - \left(E\left[X^2\right]\right)^2 \\ &= 3\sigma^4 - \sigma^4 \\ &= 2\sigma^4. \end{aligned}$$

(c) Notice that $E\left[Y_iY_jY_kY_l\right] = 0$ for pairwise distinct i, j, k, and l by independence. Also $E\left[Y_iY_j^2Y_l\right] = 0$ for $i \neq j \neq l$, and $E\left[Y_i^3Y_j\right] = 0$ for $i \neq j$. Therefore,

$$\begin{aligned} E\left[X_n^4\right] &= \frac{1}{n^2}E\left[\left(\sum_{i=1}^n Y_i\right)\left(\sum_{j=1}^n Y_j\right)\left(\sum_{k=1}^n Y_k\right)\left(\sum_{l=1}^n Y_l\right)\right] \\ &= \frac{3}{n^2}\sum_{i=1}^n\sum_{j=1}^n E\left[Y_i^2Y_j^2\right] + \frac{1}{n^2}\sum_{i=1}^n E\left[Y_i^4\right] \\ &= 3n(n-1)\sigma^4 + \frac{\mu_4}{n}. \end{aligned} \tag{1}$$

(d) As $n \to \infty$, Eq (1) gives that $E\left[X_n^4\right] \to 3\sigma^4$. Thus the central limit theorem says that the influence of μ_4 should disappear in the limit as $n \to \infty$.

2. **(a)** If X_i is replaced by $X_i - \mu$, then

$$\begin{aligned} \hat{\mu}_{\text{new}} &= \frac{1}{n}\sum_{i=1}^n (X_i - \mu) \\ &= \hat{\mu}_{\text{old}} - \mu. \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{\sigma^2} &= \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \bar{X}_{\text{new}}\right)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \mu - \bar{X}_{\text{old}} + \mu\right)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \bar{X}_{\text{old}}\right)^2. \end{aligned}$$

(b) If X_i is replaced by $\frac{1}{\sigma}X_i$, then

$$\begin{aligned}\hat{\mu}_{\text{new}} &= \frac{1}{n\sigma}\sum_{i=1}^{n} X_i \\ &= \frac{1}{\sigma}\hat{\mu}_{\text{old}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widehat{\sigma^2} &= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{1}{\sigma}X_i - \bar{X}_{\text{new}}\right)^2 \\ &= \frac{1}{n-1}\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \hat{\mu}_{\text{old}}\right)^2 \\ &= \frac{1}{n-1}\frac{\widehat{\sigma^2}}{\sigma^2}.\end{aligned}$$

(c)

$$\|v_1\|_{l^2} = \left(\sum_{i=1}^{n}\frac{1}{n}\right)^{\frac{1}{2}} = 1.$$

Every finite-dimensional inner product space has an orthonormal basis, which may be obtained from an arbitrary basis using the Gram–Schmidt process.

(d) Given $v \in \mathbb{R}^n$ and $\mathcal{S} \in \mathbb{R}^n$ is be the plane of vectors perpendicular to v. Let $x \in \mathbb{R}^n$ such that $y \in \mathcal{S}$ is the orthogonal projection of x onto $\mathcal{S}$, then we must have the minimum

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2x^t y,$$

which means $x^t y = \|y\|^2$. Therefore, since v is perpendicular to y, we can decompose x as

$$\begin{aligned}x &= (x^t v)\frac{v}{\|v\|^2} + (x^t y)\frac{y}{\|y\|^2} \\ &= (x^t v)v + y, \end{aligned} \tag{2}$$

which asserts **i.**.

From Eq. (2), $x - y = (x^t v)v$ is perpendicular to $x - y$. So **ii.** is true. To verify **iii**, we square Eq. (2),

$$\begin{aligned}\|x\|^2 &= (x^t v)^2 + \|y\|^2 + 2(x^t v)v^t y \\ &= (x^t v)^2 + \|y\|^2\end{aligned}$$

from **ii.**.

(e) Trivial computation,

$$\bar{X} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i = \frac{1}{\sqrt{n}}X^t v_1.$$

(f) Note that

$$\begin{aligned} E\left[Y_i \bar{X}\right] &= \frac{1}{\sqrt{n}} E\left[\sum_{j=1}^{n} v_{i,j} X_j \sum_{k=1}^{n} X_k\right] \\ &= E\left[Y_i\right] E\left[\bar{X}\right] \end{aligned}$$

by the independence of X. The same reason we prove that the independence of Y_i. Also the orthogonality gives the fact $Y_i \sim \mathcal{N}(0,1)$.

(g) Let Y be the orthogonal projection of X onto S. Then

$$\begin{aligned} \|Y\|^2 &= \sum_{i=2}^{n} Y_i^2 \\ \text{(by 2(d) i)} &= \sum_{i=1}^{n} \left(X_i - \left[X^t v_1\right] v_{1,i}\right)^2 \\ &= \sum_{i=1}^{n} \left(X_i - \sqrt{n}\bar{X} \frac{1}{\sqrt{n}}\right)^2 \\ &= \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 . \end{aligned}$$

(h) Notice from **(g)**, since

$$\widehat{\sigma^2} = \frac{1}{n-1} \sum_{i=2}^{n} Y_i^2,$$

a function of $Y_2, Y_3, \cdots, Y_n$, and $\hat{\mu}$ is a function of Y_1 only. They must be independent to each other.

(i) From part **(f)** we have $Y_i \sim \mathcal{N}(0,1)$, therefore

$$\begin{aligned} \widehat{\sigma^2} &= \frac{1}{n-1} \sum_{i=2}^{n} Y_i^2 \\ &\sim \frac{1}{n-1} \chi^2_{n-1}. \end{aligned}$$

Also,

$$\begin{aligned} E[\widehat{\sigma^2}] &= \frac{1}{n-1} \sum_{i=2}^{n} E[Y_i^2] \\ &= \frac{n-1}{n-1} = 1. \end{aligned}$$

(j)

3. (a) Consider the $n-1$ samples x_i to represent a single point in a $(n-1)-$dimensional space. The chi squared distribution for $n-1$ degrees of freedom will then be given by the prodoct of Gaussian $\mathcal{N}(0,1)$,

$$\begin{aligned} F(q) &= \mathbf{Pr}(Q \le q) \\ &= \frac{1}{(2\pi)^{(n-1)/2}} \int_{\|\mathbf{x}\|^2 \le q} e^{-\|\mathbf{x}\|^2/2} dx_1 dx_2 \ldots dx_{n-1}. \end{aligned}$$

Let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$, since it is a constant, it may be removed from inside the integral first

$$F(q) = \frac{e^{-r^2/2}}{(2\pi)^{(n-1)/2}} \int_{\mathcal{S}} dx_1 dx_2 \ldots dx_{n-1}. \tag{3}$$

The integral in (3) is now simply the surface area A of the $n-2$ sphere times the infinitesimal thickness of the sphere which is $dr = \frac{dq}{2r}$. The area of a $n-2$ sphere is

$$A = \frac{(n-1)r^{n-2}\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)}.$$

Substituting and cancelling terms yields,

$$\begin{aligned} F(q) &= \frac{e^{-r^2/2}}{(2\pi)^{(n-1)/2}} \int_{r=0}^{\sqrt{q}} \frac{(n-1)r^{n-2}\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} 2r dr \\ &= \underbrace{\frac{2\pi^{\frac{n-1}{2}}(n-1)}{\Gamma(\frac{n-1}{2}+1)(2\pi)^{(n-1)/2}}}_{\text{constant}} \int_{r=0}^{\sqrt{q}} e^{-r^2/2} r^{n-2} r dr \\ &= C \int_{r=0}^{\sqrt{q}} r^{n-2} e^{-r^2/2} dr. \end{aligned}$$

The second part is nothing but differentiating $F(q)$ with respect to q

$$\begin{aligned} f(q) &= F'(q) \\ &= C q^{-\frac{1}{2}} q^{\frac{n-1}{2}} e^{-\frac{q}{2}} \\ &= C q^{\frac{n-2}{2}} e^{-\frac{q}{2}}. \end{aligned}$$

(b) $|t| \le x$ implies

$$\begin{aligned} \left| \frac{\sqrt{n-1}Z}{\sqrt{Q}} \right| &\le x \\ |Z| &\le \frac{x\sqrt{Q}}{\sqrt{n-1}}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
Pr(|t| < x) =& Pr(|Z| \leq \frac{x\sqrt{Q}}{\sqrt{n-1}}) \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\frac{x\sqrt{Q}}{\sqrt{n-1}}}^{\frac{x\sqrt{Q}}{\sqrt{n-1}}} e^{-\frac{z^2}{2}} dz \\
=& \frac{C}{\sqrt{2\pi}} \int_0^\infty q^{\frac{n-2}{2}} e^{-\frac{q}{2}} \int_{-\frac{x\sqrt{q}}{\sqrt{n-1}}}^{\frac{x\sqrt{q}}{\sqrt{n-1}}} e^{-\frac{z^2}{2}} dz dq. \qquad (4)
\end{aligned}
$$

(c) Differentiate Eq.(4) with respect to x,

$$
f(x) = \frac{C}{\sqrt{\frac{1}{2}(n-1)\pi}} \int_0^\infty q^{\frac{n-1}{2}} e^{-\frac{q}{2}} e^{-\frac{x^2 q}{2(n-1)}} dq.
$$

(d) Let $a(x)q = \frac{1}{2}\left(1 + \frac{1}{n-1}x^2\right) q = r$, then

$$
\begin{aligned}
f(x) &= C \int_0^\infty q^{\frac{n-1}{2}} e^{-a(x)q} dq \\
&= 2C \int_0^\infty \left(1 + \frac{1}{n-1}x^2\right)^{-\frac{n-1}{2}} \frac{1}{2}\left(1 + \frac{1}{n-1}x^2\right) r^{\frac{n-1}{2}} e^{-r} dr \\
&= \frac{C}{\left(1 + \frac{1}{n-1}x^2\right)^{\frac{n}{2}}} \int_0^\infty r^{\frac{n-1}{2}} e^{-r} dr \\
&= \frac{C}{\left(1 + \frac{1}{n-1}x^2\right)^{\frac{n}{2}}} \Gamma\left(\frac{n+1}{2}\right) \\
&= \frac{C}{\left(1 + \frac{1}{n-1}x^2\right)^{\frac{n}{2}}}.
\end{aligned}
$$

(e) Consider the simplest case $\mu = 0$, $\sigma = 1$,

$$
\begin{aligned}
E[X^2] &= C \int_{\mathbb{R}} \frac{x^2 dx}{\left(1 + \frac{x^2}{n}\right)^{\frac{n}{2}+1}} \\
&= \frac{1}{\sqrt{\pi \Gamma\left(\frac{n}{2}\right)}} \left[\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{n-2}{2}\right) n\right].
\end{aligned}
$$

Then for $n = 3$, we have

$$
\begin{aligned}
E[X^2] &= \frac{1}{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)} \left[\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right) 3\right] \\
&= \frac{1}{\sqrt{\pi}\frac{1}{2}\sqrt{\pi}} \left[\frac{1}{2}\sqrt{\pi}\sqrt{\pi}3\right] \\
&= 3 \neq 1 < \infty.
\end{aligned}
$$

(f) To show the power law tail, we just look at the asymptotic behavior of $f(x)$

$$\begin{aligned} f(x) &= \frac{C}{\left(1+\frac{(x-\mu)^2}{n\sigma^2}\right)^{\frac{n}{2}+1}} \\ &\approx C(1+x^2)^{-(n/2+1)} \\ &\approx Cx^{-p}. \end{aligned}$$

4. Rewrite in matrix form

$$\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{16} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} Z_n.$$

The eigenvalues of matrix A are $\frac{1}{4} \pm \frac{1}{2}i$, and thus $|\lambda| < 1$ that the Gaussian process is stable.