Assignment 3, due October 1

Corrections: (none yet.)

1. (*Time change*) This exercise gives a way to turn a Brownian motion into an Ornstein Uhlenbeck process. Suppose W_t is a standard Brownian motion with $\mathrm{E}[\Delta W|\mathcal{F}_t] = 0$ and $\mathrm{E}\big[\Delta W^2|\mathcal{F}_t\big] = \Delta t$. Here $\Delta W = W_{t+\Delta t} - W_t$, and $\Delta t > 0$. The distribution of W_t has width approximately $\sqrt{t}$, which grows as $t \to \infty$ and goes to zero as $t \to 0$. The Ornstein Uhlenbeck process has a statistical steady state, so its width is not suppose to go to infinity or zero as $t \to \infty$. A function with width $(t^{-1/2} \cdot \sqrt{t})$, such as $Y_t = t^{-1/2} W_t$, should approximately independent of t. A *time change* is a function $t = t(s)$ with $t'(s) = \partial_s t(s) > 0$ when $t > 0$. A time change applied to Y_t gives $X_s = Y_{t(s)}$.

   (a) Define $\Delta Y = Y_{t+\Delta t} - Y_t$ with $\Delta t > 0$. Find formulas for μ_Y and σ_Y^2 so that $\mathrm{E}[\Delta Y|\mathcal{F}_t] = \mu_Y \Delta t +$ (smaller) and $\mathrm{E}\big[\Delta Y^2|\mathcal{F}_t\big] = \sigma_Y^2 \Delta t +$ (smaller). By "(smaller)", we mean something that goes to zero faster than Δt, as $\Delta t \to 0$. For example, $\sqrt{t + \Delta t} - \sqrt{t} = \frac{1}{2\sqrt{t}} \Delta t + O(\Delta t^2)$. Here, $f(\Delta t) = O(\Delta t^2)$ means that there is a C and an ϵ so that if $\Delta t \leq \epsilon$, then $f(\Delta t) \leq C \Delta t^2$. This is the " *big O*" notation, and $O(\Delta t^2)$ is read: "on the order of Δt^2". For this problem, (smaller) is the same as $O(\Delta t^2)$. The errors in Taylor series are like this most of the time. This is the mathematicians' idea of order, which refers to scaling rather than size.

Sol: Let us consider the infinitesimal mean first

$$\begin{aligned}
E[\Delta Y|\mathcal{F}_t] &= E[Y_{t+\Delta t} - Y_t|\mathcal{F}_t] \\
&= E\left[\frac{1}{\sqrt{t+\triangle t}} W_{t+\triangle t} - \frac{1}{\sqrt{t}} W_t \middle| \mathcal{F}_t \right] \\
&= E\left[\frac{1}{\sqrt{t+\triangle t}} (W_{t+\triangle t} - W_t) + \left(\frac{1}{\sqrt{t+\triangle t}} - \frac{1}{\sqrt{t}} \right) W_t \middle| \mathcal{F}_t \right] \\
&= \frac{1}{\sqrt{t+\triangle t}} \underbrace{E[(W_{t+\triangle t} - W_t)| \mathcal{F}_t]}_{=0} + E\left[\left(\frac{1}{\sqrt{t+\triangle t}} - \frac{1}{\sqrt{t}} \right) W_t \middle| \mathcal{F}_t \right] \\
&= E\left[\left(\frac{1}{\sqrt{t+\triangle t}} - \frac{1}{\sqrt{t}} \right) W_t \middle| \mathcal{F}_t \right] \\
&= E\left[\left(-\frac{1}{2} t^{-\frac{3}{2}} \triangle t + \frac{3}{8} (\triangle t)^2 + \mathcal{O}((\triangle t)^3) \right) W_t \middle| \mathcal{F}_t \right] \\
&= E\left[\left(-\frac{1}{2} t^{-1} Y_t \triangle t + \frac{3}{8} t^{-\frac{5}{2}} (\triangle t)^2 W_t + \mathcal{O}((\triangle t)^3) W_t \right) \middle| \mathcal{F}_t \right] \\
&= -\frac{1}{2t} Y_t \triangle t + \mathcal{O}(\triangle t^2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{E}\left[\Delta Y^2|\mathcal{F}_t\right] &= \mathrm{E}\left[\left(\frac{1}{\sqrt{t}} (W_{t+\triangle t} - W_t) + \underbrace{\left(\frac{1}{\sqrt{t+\triangle t}} - \frac{1}{\sqrt{t}} \right)}_{=-\frac{1}{2} t^{-\frac{3}{2}} \triangle t + \frac{3}{8} (\triangle t)^2} W_{t+\triangle t} \right)^2 \middle| \mathcal{F}_t \right] \\
&= \mathrm{E}\left[\frac{1}{t} (W_{t+\triangle t} - W_t)^2 - 2\frac{1}{2} t^{-\frac{5}{2}} (W_{t+\triangle t} - W_t) W_{t+\triangle t} \triangle t + \mathcal{O}(\triangle t^2) \middle| \mathcal{F}_t \right] \\
&= \frac{1}{t} \triangle t - 2\frac{1}{2} t^{-\frac{5}{2}} \triangle t \underbrace{\mathrm{E}[W_{t+\triangle t} - W_t| \mathcal{F}_t]}_{=0} \mathrm{E}[W_{t+\triangle t}| \mathcal{F}_t] + \mathcal{O}(\triangle t^2) \\
&= \frac{1}{t} \triangle t + \mathcal{O}(\triangle t^2).
\end{aligned}$$

(b) (*Please do this before the next part*) Speculate on the sizes ΔY_t for fixed Δt and large t. Is Y_t changing rapidly or slowly for large t? The rate of change of the Ornstein Uhlenbeck process is roughly constant for all time. If we want X_s to be an Ornstein Uhlenbeck process, should we look for $\Delta t \gg \Delta s$ or $\Delta t \ll \Delta s$?

Sol: Since $Y_t \sim \sqrt{t} \cdot \frac{1}{\sqrt{t}} = C$, for some constant C, Y_t changes slowly for large t. If we fix $\triangle t$ and let t large, then the change of Y_t, namely $\triangle Y$ has mean and vairance both 0. Supposed that $\triangle t \ll \triangle s$, then $t'(s) \approx \frac{\triangle t}{\triangle s} \approx 0$, which means $t(s)$ is constant as s changes. So $X_s = Y_{t(s)=c} \approx C$. Therefore, if we want X_s to be an Ornstein Uhlenbeck process, we must look for $\triangle t \gg \triangle s$.

(c) Write a formula for σ_X^2 in

$$\begin{aligned}\mathrm{E}\left[(X_{s+\Delta s} - X_s)^2 \middle| \mathcal{F}_t \right] &= \mathrm{E}\left[\left(Y_{t(s)+\Delta t(\Delta s)} - Y_{t(s)}\right)^2 \middle| \mathcal{F}_t \right] \\ &= \Delta s \sigma_X^2 + \text{(smaller)} \\ &= \Delta t \sigma_Y^2 + \text{(smaller)}\end{aligned}$$

in terms of Δs and $\Delta t = t(s+\Delta s) - t(s) \approx t'(s)\Delta s$. Find and solve the differential equation for $t(s)$ so that σ_X^2 does not depend on s. Does your quantitative solution here agree with the qualitative guess from part (b)?

Sol:

$$\begin{aligned}E\left[(X_{s+\Delta s} - X_s)^2 \middle| \mathcal{F}_t \right] &= E\left[\left(Y_{t(s)+\Delta t(\Delta s)} - Y_{t(s)}\right)^2 \middle| \mathcal{F}_t \right] \\ &= \frac{1}{t(s)} \triangle t(\Delta s) + \mathcal{O}(\triangle t^2) \\ &= \frac{1}{t(s)} t'(s)\Delta s + \mathcal{O}(\Delta s^2).\end{aligned}$$

Assume that σ_X^2 is independent to s, namely

$$\frac{1}{t(s)} t'(s) = c > 0,$$

since by assumption $t'(s) > 0$, which gives $t = e^{cs}$. So $\triangle t$ indeed $\gg \triangle s$.

To see how Δt and Δs are related, note that

$$\begin{aligned}\Delta t &= t(s+\Delta s) - t(s) \\ &= e^{c(s+\Delta s)} - e^{cs} \\ \ln(t+\Delta t) - \ln t &= c(s+\Delta s) - cs \\ &= c\Delta s \\ \Delta s &= \frac{1}{c} \ln\left(\frac{t+\Delta t}{t}\right).\end{aligned}$$

(d) With the time change from part (c), find a formula for $\mu_X(X_s)$ so that $\mathrm{E}[X_{s+\Delta s} - X_s | \mathcal{F}_s] = \mu_X(X_s)\Delta s + \text{(smaller)}$. Does this imply that X_s is an Ornstein Uhlenbeck process?

Sol: Consider

$$\begin{aligned}\mathrm{E}\left[X_{s+\Delta s} - X_s \middle| \mathcal{F}_t \right] &= -\frac{1}{2t(s)} X_{t(s)} t'(s) \triangle s + \mathcal{O}(\triangle t^2) \\ &= -\frac{c}{2} X_s \triangle s + \mathcal{O}(\triangle t^2).\end{aligned}$$

An Ornstein–Uhlenbeck process, X_t, satisfies the following stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

where $\theta > 0$, μ and $\sigma > 0$ are parameters and denotes W_t the Wiener process. The above representation can be taken as the primary definition of an Ornstein–Uhlenbeck process. Here we know from part (c), $\mathrm{E}\left[(dX_s)^2\right] = \text{constant} \cdot ds$. Also

2. (*Moving toward Ito's lemma*) This exercise explores some of the ideas that lead to Ito's lemma. The full Ito lemma asks you to take a large number of time steps. This exercise this week explains what happens in one step.

   (a) Suppose $Y \sim \mathcal{N}(0, \epsilon)$. Find the scaling laws for $m_k = \mathrm{E}\left[|Y|^k\right]$.
   Hint #1 (*You don't need to do it both ways*): Write the probability density for Y and do a change of variables to make the density independent of ϵ, then see how ϵ comes out of the expectation integral. It will be a power of ϵ.
   Hint #2: (*the easier way, but really equivalent*) Write $Y = \epsilon^p Z$ where $Z \sim \mathcal{N}(0, 1)$, and see what power of ϵ you need to get the right Y distribution. Then $m_k = \epsilon^{r_k} \mathrm{E}\left[|Z|^k\right]$, where the expectation now is just a number that does not depend on ϵ. Write the explicit formula for m_4, which we saw in assignment 1.
   Sol: #1. Straightforward computation letting $y/\sqrt{2\epsilon} = z$,

$$\begin{aligned}
m_k &= \frac{1}{\sqrt{2\pi\epsilon}} \int_{-\infty}^{\infty} |y|^k e^{-\frac{y^2}{2\epsilon}} dy \\
&= \frac{1}{\sqrt{2\pi\epsilon}} \int_{-\infty}^{\infty} \sqrt{2\epsilon} \left|\sqrt{2\epsilon} z\right|^k e^{-z^2} dz \\
&= \frac{(2\epsilon)^{k/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} |z|^k e^{-z^2} dz \\
&= \frac{(2\epsilon)^{k/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{2} u^{-\frac{1}{2}} |u|^{\frac{k}{2}} e^{-u} du \\
&= \frac{(2\epsilon)^{k/2}}{\sqrt{\pi}} \int_{0}^{\infty} u^{\frac{k+1}{2}-1} e^{-u} du \\
&= \frac{(2\epsilon)^{k/2}}{\sqrt{\pi}} \Gamma(\frac{k+1}{2}) \\
&= \epsilon^{\frac{k}{2}} (k-1)(k-3) \cdots
\end{aligned}$$

   withe the odd moments all are zero. Therefore,

$$\begin{aligned}
m_4 &= \frac{(2\epsilon)^{4/2}}{\sqrt{\pi}} \Gamma(\frac{4+1}{2}) \\
&= \frac{4\epsilon^2}{\sqrt{\pi}} \frac{3}{4} \sqrt{\pi} \\
&= 3\epsilon^2.
\end{aligned}$$

(b) Suppose $f(w)$ is a differentiable function, and $X_t = f(W_t)$, with W_t being a standard Brownian motion. Find formulas for μ_X and σ_X^2 so that

$$\mu_X(W_t)\Delta t = \mathrm{E}[\Delta X \mid \mathcal{F}_t] + (\text{smaller})$$
$$\sigma_X^2(W_t)\Delta t = \mathrm{E}\big[\Delta X^2 \mid \mathcal{F}_t\big] + (\text{smaller}) \ .$$

The formulas for μ and σ^2 will depend on $f'(W_t)$ and $f''(W_t)$. You will need to use the result of part (a) to justify not needing more derivatives of f. You need to show that contributions from these terms are (smaller).

Sol: Notice that from Ito,

$$dX_t = f'(X_t)dW_t + \frac{1}{2}f''(X_t)dt.$$

Thus

$$\begin{aligned}
\mathrm{E}\left[\triangle X \mid \mathcal{F}_t\right] &= \mathrm{E}\left[X_{t+\triangle t} - X_t \mid \mathcal{F}_t\right] \\
&= \mathrm{E}\left[f(W_{t+\triangle t}) - f(W_t) \mid \mathcal{F}_t\right] \\
&= \mathrm{E}\left[f'\left(W_t\right)\left(W_{t+\triangle t} - W_t\right) + \frac{1}{2!}f''\left(W_t\right)\left(W_{t+\triangle t} - W_t\right)^2 + h.o.t \mid \mathcal{F}_t\right] \\
&= \frac{1}{2!}f''\left(W_t\right)\triangle t + \mathcal{O}(\triangle t^2).
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{E}\left[\triangle X^2 \mid \mathcal{F}_t\right] &= \mathrm{E}\left[\left(f(W_{t+\triangle t}) - f(W_t)\right)^2 \middle| \mathcal{F}_t\right] \\
&= \mathrm{E}\left[\left(f'\left(W_t\right)\right)^2\left(W_{t+\triangle t} - W_t\right)^2 + h.o.t \middle| \mathcal{F}_t\right] \\
&= \left(f'\left(W_t\right)\right)^2\triangle t + \mathcal{O}(\triangle t^2).
\end{aligned}$$

3. (*Integrals of Brownian motion*) Some integrals involving Brownian motion can be done just by calculating means and variances. Suppose $W_{[0,T]}$ is a standard Brownian motion path up to time t. Define

$$X = \int_0^T t^2 W_t\, dt \ .$$

This X is a linear function of the Gaussian Brownian motion path so X is a Gaussian random variable. You determine the distribution of X completely by determining its mean and variance.

(a) Find a formula for $\mathrm{cov}(W_t, W_s) = \mathrm{E}[W_t W_s]$. Hint: Assume at first that $t > s$ and write $W_t = W_s + (W_t - W_s)$, then use the independent

increments property.
Sol: Suppose that $t > s$,

$$\begin{aligned} cov(W_t, W_s) &= E\left[W_t W_s\right] \\ &= E\left[W_s^2 + (W_t - W_s)W_s\right] \\ &= E\left[W_s^2\right] + E\left[(W_t - W_s)\right] \underbrace{E\left[W_s\right]}_{=0} \\ &= s. \end{aligned}$$

Same trick we must have $\text{cov}(W_t, W_s) = t$ for $t < s$; namely $\text{cov}(W_t, W_s) = \min(t, s)$.

(b) Find $\text{var}(X)$. Write $X^2 = \int_0^T t^2 W_t dt \int_0^T s^2 W_s ds = \int_0^T \int_0^T t^2 s^2 W_t W_s dt ds$. $\mathrm{E}\left[X^2\right]$ now is the expectation of a double integral, and you can take the expectation inside the integral and use the result of part (a).
Sol: Correction,

$$\begin{aligned} \text{var}(X) =& \mathrm{E}\left[X^2\right] - \left(\mathrm{E}\left[X\right]\right)^2 \\ =& \int_0^T \int_0^T t^2 s^2 \mathrm{E}\left[W_t W_s\right] dt ds - \left(\int_0^T t^2 W_t dt\right)^2 \\ =& \int_0^T \int_0^s t^3 s^2 dt ds + \int_0^T \int_s^T t^2 s^3 dt ds - 0 \\ =& \int_0^T s^2 \int_0^s t^3 dt ds + \int_0^T s^3 \int_s^T t^2 dt ds \\ =& \frac{1}{4}\frac{T^7}{7} + \frac{1}{3}\left(\frac{T^7}{4} - \frac{T^7}{7}\right) \\ =& \frac{T^7}{14}. \end{aligned}$$

Previous version,

$$
\begin{aligned}
\mathrm{var}\left(X^2\right) =&\mathrm{E}\left[X^4\right] - \left(\mathrm{E}\left[X^2\right]\right)^2 \\
=&\int_0^T \int_0^T t^4 s^4 \mathrm{E}\left[W_t W_s\right] dtds - \left(\int_0^T \int_0^T t^2 s^2 \mathrm{E}\left[W_t W_s\right] dtds\right)^2 \\
=&\int_0^T \int_0^s t^5 s^4 dtds + \int_0^T \int_s^T t^4 s^5 dtds - \left(\int_0^T \int_0^s t^3 s^2 dtds + \int_0^T \int_s^T t^2 s^3 dtds\right)^2 \\
=&\frac{1}{6}\int_0^T s^{10} ds + \frac{1}{5}\int_0^T s^5 \left(T^5 - s^5\right) ds - \left(\int_0^T s^2 \int_0^s t^3 dtds + \int_0^T s^3 \int_s^T t^2 dtds\right)^2 \\
=&\frac{1}{6}\frac{T^{11}}{11} + \frac{1}{5}\left(\frac{T^{11}}{6} - \frac{T^{11}}{11}\right) - \left(\frac{1}{4}\frac{T^7}{7} + \frac{1}{3}\left(\frac{T^7}{4} - \frac{T^7}{7}\right)\right)^2 \\
=&\frac{T^{11}}{33} - \left(\frac{T^7}{14}\right)^2 .
\end{aligned}
$$

4. (*computing*) This assignment explores the properties of Brownian Motion and the Ornstein Uhlenbeck process via simulation. It also introduces you to the slowness of Monte Carlo simulation in general. You have to push the computer pretty hard to get good looking plots. The slowness of `R` does not help.

   This program generates L sample Brownian motion or Ornstein Uhlenbeck paths, all independent, each with time step $dt = T/N$, where T is the end of the time interval and N is the number of time steps, so $t_N = T$. Let $W_{[0,T]}$ be a path, and $Y = F(w_{[0,T]})$ a function of the path. This assignment just makes histograms of the distributions of various path functionals.

   (a) Download the files `Assignment3.R` and `AssignmentStart3.pdf`. If you run the `R` program "out of the box" (exactly as downloaded, all parameters unchanged), you should get a picture that looks like the picture, possibly with different noise. Actually, you need to change one parameter, the name of the directory for the output plot `.pdf` file. This makes a histogram of

   $$F(W_{[0,T]}) = M_T = \max_{0 \leq t \leq T} W_t \ .$$

   The picture `Assignment3.pdf` is a normalized histogram that estimates the probability density, $f(m)$, of the random variable M_T. Next week we will use the *Kolomogorov reflection principle* to find a formula for $f(m)$. This week's picture agrees with next week's formula, hopefully.

   (b) The out-of-the-box picture is not very clear. Try to make it clearer by turning up the computational parameters N and L. Larger N

reduces the spurious high value at $m = 0$ and "rounds out" the rest of the picture. We will learn later how it does this. Larger L reduces statistical noise so you can see the curve more clearly. Do this until the run time is more than you have patience for. You will learn two of the prime drawbacks of Monte Carlo: it is slow; it is noisy.

(c) The code out-of-the-box has partly implemented the study of the distribution of W_T conditional on $W_t \leq B$ for all $t \in [0, T]$. Use the $T = 20$. If you simulate Brownian motion and only count paths that do not touch a *barrier*, you are simulating Brownian motion with *an absorbing boundary* at B. Modify the `R` code to estimate the probability density of W_T conditional on not hitting B before T. You will have to change the parameter `bs` (the *starting bin*) and possibly other things having to do with generating the histogram. Push the computation to get the clearest picture you can with the computer and time constraints you have. Next week we will find a formula for this distribution.

(d) Modify the program to simulate the Ornstein Uhlenbeck process with $\sigma = 2$ and $\gamma = .1$. Start with $X_0 = 2$. On one plot, put the distributions of X_T for $T = 5$ and $T = 10$, both the normalized histograms and the exact formulas for the probability densities. Copy some code from `Assignment2.R` to put multiple plots in the same frame. The exact formulas are from class.