Stochastic Calculus, Courant Institute, Fall 2012
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2012/index.html
Always check the class message board on the blackboard site from home.nyu.edu before doing any work on the assignment.

Assignment 4, due October 7

Corrections: (none yet)

1. (*Brownian motion with reflection*) A *reflecting Brownian motion*, with a *reflecting barrier* at $x = a$, is a stochastic process that never crosses a and does not stick to a. For $X_t \neq a$, X_t acts like a Brownian motion. Suppose $X_0 = 0$ and $a > 0$. A reflecting Brownian motion has a probability density, $X_t \sim u(x,t)$, that satisfies the heat equation if $x < a$, and has

$$\int_{-\infty}^{a} u(x,t)\,dx = 1 \; . \tag{1}$$

(a) The conservation formula (1) implies a boundary condition that u satisfies at $x = a$. What is this condition? Hint: What must the probability flux be at $x = a$? This boundary condition is called a *reflecting* boundary condition. For `wikipedia` lovers, it is also called a *Neumann* boundary condition.

Sol: Differentiating $u(x,t)$ with respect to time t and using the heat equation,

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{a} u(x,t)dx &= \int_{-\infty}^{a} u_t(x,t)dx \\ &= \frac{1}{2}\int_{-\infty}^{a} u_{xx}(x,t)dx \\ &= \frac{1}{2}\left(u_x(a,t) - u_x(-\infty,t)\right) \\ &= 0. \end{aligned}$$

Since the heat kernel goes to 0 as $x \to -\infty$, we have found the *Neumann* boundary condition

$$u_x(a,t) = 0.$$

(b) Show that if $v(x)$ is symmetric about the point a, which is the condition $v(a-x) = v(a+x)$ for all x, and if v is a smooth function of x, then v satisfies the boundary condition from part a.

Sol: If $v(x)$ is symmetric about a, then $v(a-x) = v(a+x)$ and thus

$$\partial_x v(a-x) = v'(a-x)\cdot(-1) = v'(a+x)\cdot 1 = \partial_x v(a+x).$$

Substituting $x = 0$ into above equation, $-v'(a) = v'(a)$, which implies

$$v_x(a) = 0.$$

(c) Use the method of images from this week's material to write a formula for the $u(x,t)$ that satisfies the correct initial condition for $X_0 = 0$ and boundary condition at $a > 0$. It is closely related to the formula from class.

Sol: We want a function $u(x,t)$ that is defined for $x < a$ that satisfies the initial condition $u(x,t) \to \delta(x)$ as $t \to 0$, for $(x < a)$ and the reflexing boundary condition $u_x(a,t) = 0$. Since from (b) we know if $v(x)$ symmetric about a, then v satisfies this boundary condition. We extend the definition of u by symmetric,

$$u(a+x,t) = u(a-x,t),$$

or saying $x' = 2a - x$ and we have

$$u(x',t) = u(x,t).$$

The resulting initial data becomes

$$u(x,0) = \delta(x) + \delta(2a - x).$$

The initial data has changed, but the part for $x < a$ is the same. The solution is the superposition of the pieces from the two delta functions:

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-x^2/2t} + e^{-(2a-x)^2/2t} \right).$$

Notice that from notes

$$\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \int_{-\infty}^a \frac{1}{\sqrt{2\pi t}} e^{-(x-2a)^2/2t} dx,$$

therefore

$$\begin{aligned}
\int_{-\infty}^a u(x,t)dx &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^a \left(e^{-x^2/2t} + e^{-(2a-x)^2/2t} \right) dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^a e^{-x^2/2t} dx + \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^a e^{-(x-2a)^2/2t} dx \\
&= \int_{-\infty}^a \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx + \int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx \\
&= 1.
\end{aligned}$$

Also the Neumann boundary condition,

$$u_x(a,t) = \frac{1}{\sqrt{2\pi t}}\left(-\frac{2x}{2t}e^{-x^2/2t} + \frac{2(2a-x)}{2t}e^{-(2a-x)^2/2t}\right)\Bigg|_{x=a}$$
$$= \frac{1}{\sqrt{2\pi t}}\left(-\frac{a}{t}e^{-a^2/2t} + \frac{a}{t}e^{-a^2/2t}\right) = 0$$

(d) Write a formula for $m_t = \mathrm{E}[\,X_t]$ for reflecting Brownian motion. The *cumulative normal* distribution is $N(z) = \mathrm{P}(Z \le z)$, when $Z \sim \mathcal{N}(0,1)$. Derive a formula for m_t in terms of this and other explicit functions. Verify that m_t is exponentially small for small t. Verify that $m_t \to \infty$ as $t \to \infty$ and scales as $t^{1/2}$.

Sol: Consider the change of variables, $z_1 = x/\sqrt{t}$ and $z_2 = (x - 2a)/\sqrt{t}$, then

$$\mathrm{E}[X_t] = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a} xe^{-x^2/2t} + xe^{-(2a-x)^2/2t}dx$$
$$= \frac{\sqrt{t}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{a}{\sqrt{t}}} z_1 e^{-z_1^2/2}dz_1 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{a}{\sqrt{t}}}\left(\sqrt{t}z_2 + 2a\right)e^{-z_2^2/2}dz_2$$
$$= \frac{\sqrt{t}}{\sqrt{2\pi}}\left(-e^{-\frac{z_1^2}{2}}\Bigg|_{z_1=-\infty}^{z_1=\frac{a}{\sqrt{t}}}\right) + \frac{\sqrt{t}}{\sqrt{2\pi}}\left(-e^{-\frac{z_2^2}{2}}\Bigg|_{z_2=-\infty}^{z_2=-\frac{a}{\sqrt{t}}}\right) + \frac{2a}{\sqrt{2\pi}}N(-\frac{a}{\sqrt{t}})$$
$$= \sqrt{\frac{2}{\pi}}\left[aN(-\frac{a}{\sqrt{t}}) - \sqrt{t}e^{-\frac{a^2}{2t}}\right].$$

Notice that as $t \to \infty$,

$$m_t \to \sqrt{\frac{2}{\pi}}\left[aN(0) - \sqrt{t}\left(1 - \frac{a^2}{2t} + \mathcal{O}(t^{-2})\right)\right]$$
$$\to -\infty.$$

(e) It is argued (possibly later in this course, or the book *Stochastic Integrals* by Henry McKean) that a reflecting Brownian motion is kept inside the allowed region $\{x \le a\}$ by a rightward force at the reflecting boundary. This force is different from zero only when $X_t = a$. The force is just strong enough to prevent $X_t > a$. This picture suggests that the total force is proportional to the total time spent at the boundary. Since only the boundary force has a preferred direction, if $X_0 = 0$, it may be that

$$\mathrm{E}[\,X_t] = \mathrm{E}\left[\int_0^t F_s ds\right]\ ,$$

both sides being negative. Since the force only acts when $X_t = a$, it may be plausible that $\mathrm{E}[\,F_s] = C\,u(a,s)$. Verify that this picture is true, at least as far as the formula

$$m_t = -C \int_0^t u(a,s)\,ds \ .$$

Find $C > 0$.

Sol: Supposed that

$$\begin{aligned} m_t &= -C \int_0^t \frac{1}{\sqrt{2\pi s}} \left(e^{-a^2/2s} + e^{-a^2/2s} \right) ds \\ &= \sqrt{\frac{2}{\pi}} \left[aN(-\frac{a}{\sqrt{t}}) - \sqrt{t} e^{-\frac{a^2}{2t}} \right]. \end{aligned}$$

Then differentiate above with respect to t,

$$\begin{aligned} \frac{d}{dt} m_t &= -C \frac{1}{\sqrt{2\pi t}} \left(e^{-a^2/2t} + e^{-a^2/2t} \right) \\ &= -C t^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} e^{-a^2/2t}, \end{aligned}$$

which equals to

$$\begin{aligned} \frac{d}{dt} \sqrt{\frac{2}{\pi}} \left[aN(-\frac{a}{\sqrt{t}}) - \sqrt{t} e^{-\frac{a^2}{2t}} \right] &= \sqrt{\frac{2}{\pi}} \left[a \frac{d}{dt} N(-\frac{a}{\sqrt{t}}) - \frac{d}{dt} \sqrt{t} e^{-\frac{a^2}{2t}} \right] \\ &= \sqrt{\frac{2}{\pi}} \left[a e^{-\frac{a^2}{2t}} \frac{d}{dt} (-\frac{a}{\sqrt{t}}) - \left(\frac{d}{dt} \sqrt{t} \right) e^{-\frac{a^2}{2t}} - \sqrt{t} \left(\frac{d}{dt} e^{-\frac{a^2}{2t}} \right) \right] \\ &= \sqrt{\frac{2}{\pi}} \left[a e^{-\frac{a^2}{2t}} (\frac{1}{2} a t^{-\frac{3}{2}}) - \left(\frac{1}{2} t^{-\frac{1}{2}} \right) e^{-\frac{a^2}{2t}} - t^{\frac{1}{2}} \left(t^{-2} \frac{a^2}{2} e^{-\frac{a^2}{2t}} \right) \right] \\ &= -\frac{1}{2} t^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} e^{-\frac{a^2}{2t}}. \end{aligned}$$

So $C = \frac{1}{2}$ if I did nothing wrong.

2. (*Kolomogorov reflection principle*) Let X_n be a discrete time *symmetric* random walk on the integers, positive and negative. The random walk is symmetric if $\mathrm{P}(x \to x+1) = \mathrm{P}(x \to x-1)$. Suppose the walk starts with $X_0 = 0$. Let $H_a(t) = \mathrm{P}(X_n = a \text{ for some } n \leq t)$ be the hitting probability for this discrete process. Show that if $a > 0$, then

$$\mathrm{P}(H_a(t)) = \mathrm{P}(X_t = a) + 2\mathrm{P}(X_t > a) \ . \qquad (2)$$

Hint: The discrete time version of the argument from class is rigorous.

Sol: For one dimensional symmetric random walk starting at $X_0 = 0$, we define the first hitting time

$$\tau_a = \min\{ t | \, X_t = a \} \, .$$

We consider the reflecting random walk

$$Y_t = \begin{cases} X_t & \text{if } t \le \tau_a \\ 2a - X_t & \text{if } t > \tau_a \end{cases}$$

Now consider the event $\tau_a < t$. On this event, Y_t and X_t are on opposite sides of a, unless they are both at a, and they correspond under reflection. Moreover, both processes are simple random walks, so for any $k \ge 0$,

$$\mathrm{P}\left(Y_t = a + k\right) = \mathrm{P}\left(X_t = a + k\right).$$

Note the event $X_t = a + t$ is impossible unless $\tau_a < t$. So

$$\begin{aligned} \mathrm{P}\left(Y_t = a + k\right) &= \mathrm{P}\left(X_t = a + k \text{ and } \tau_a \text{<t}\right) \\ &= \mathrm{P}\left(Y_t = a + k \text{ and } \tau_a \text{<t}\right) \\ &= \mathrm{P}\left(X_t = a - k \text{ and } \tau_a \text{<t}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{P}\left(\tau_a < t\right) &= \sum_{k=-\infty}^{\infty} \mathrm{P}\left(X_t = a + k \text{ and } \tau_a \text{<t}\right) \\ &= \sum_{k=-\infty}^{-1} \mathrm{P}\left(X_t = a + k \text{ and } \tau_a \text{<t}\right) + \mathrm{P}\left(X_t = a \text{ and } \tau_a \text{<t}\right) \\ &\quad + \sum_{k=1}^{\infty} \mathrm{P}\left(X_t = a + k \text{ and } \tau_a \text{<t}\right) \\ &= \mathrm{P}\left(X_t = a\right) + \mathrm{P}\left(X_t > a\right) + \mathrm{P}\left(X_t < a\right). \\ &= \mathrm{P}\left(X_t = a\right) + 2\mathrm{P}\left(X_t > a\right). \end{aligned}$$

3. (*Brownian bridge construction of Brownian motion*) Suppose X_t is a standard Brownian motion. Suppose $0 \le t_1 < t_2 < t_3$.

   (a) Write the two dimensional PDF of (X_{t_2}, X_{t_3}) conditional on X_{t_1}. Call it $u(x_2, x_3, s_2, s_3 | x_1)$, where x_j refers to the value of X_{t_j} and $s_1 = t_2 - t_1$ and $s_2 = t_3 - t_2$. These are the time increments between t_1 and t_2, and t_2 and t_3, respectively.

   Sol: Recall that if $\mathbf{Y} = (Y_1, Y_2)^t$ be a two dimensional Gaussian vector with covariance matrix Σ^2. Then the joint density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi \det \Sigma} e^{-\mathbf{y}^t (\Sigma)^{-1} \mathbf{y}/2}.$$

Let $\mathbf{X} = [X_{t_1}, X_{t_2}, X_{t_3}]$ with $0 \le t_1 < t_2 < t_3$. The covariance matrix,

$$\sigma^2 = \begin{bmatrix} \text{Cov}\,(X_{t_1}, X_{t_1}) & \text{Cov}\,(X_{t_1}, X_{t_2}) & \text{Cov}\,(X_{t_1}, X_{t_3}) \\ \text{Cov}\,(X_{t_2}, X_{t_1}) & \text{Cov}\,(X_{t_2}, X_{t_2}) & \text{Cov}\,(X_{t_2}, X_{t_3}) \\ \text{Cov}\,(X_{t_3}, X_{t_1}) & \text{Cov}\,(X_{t_3}, X_{t_2}) & \text{Cov}\,(X_{t_3}, X_{t_3}) \end{bmatrix}$$
$$= \begin{bmatrix} t_1 & t_1 & t_1 \\ t_1 & t_2 & t_2 \\ t_1 & t_2 & t_3 \end{bmatrix}.$$

The determinant is $t_1(t_2 - t_1)(t_3 - t_2) = t_1 s_1 s_2$. The inverse is horrible. Clearly, this is not the ideal way to proceed. Let's see if we can do otherwise. The two dimensional density of (X_{t_2}, X_{t_3}) conditional on X_{t_1} can be determined from the fact the distribution of $\mathbf{Y}_1 \sim \mathcal{N}(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_1})$ conditional on $\mathbf{Y}_2\mathcal{N}(\boldsymbol{\mu_2}, \boldsymbol{\Sigma_2})$ is multivariate normal $(\mathbf{Y}_1|\mathbf{Y_2} = \mathbf{y_2}) \sim \mathcal{N}(\overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\Sigma}})$ where

$$\overline{\boldsymbol{\mu}} = \boldsymbol{\mu_1} + \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}(\mathbf{y}_2 - \boldsymbol{\mu}_2)$$
$$\overline{\Sigma} = \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}}$$

Therefore, the covariance matrix

$$\text{E}\left[X_{t_2}, X_{t_3} \middle| X_{t_1} = x_1\right] = \begin{bmatrix} \text{E}X_{t_2} \\ \text{E}X_{t_3} \end{bmatrix} + \begin{bmatrix} \text{Cov}\,(X_{t_2}, X_{t_1}) \\ \text{Cov}\,(X_{t_3}, X_{t_1}) \end{bmatrix} \frac{1}{\text{E}X_{t_1}^2}\left(x_1 - \text{E}X_{t_1}\right)$$
$$= \begin{bmatrix} \text{E}X_{t_2} \\ \text{E}X_{t_3} \end{bmatrix} + \begin{bmatrix} t_1 \\ t_1 \end{bmatrix} \frac{1}{t_1}\left(x_1 - \text{E}X_{t_1}\right)$$
$$= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}.$$

Also, the variance is

$$\text{V}\left[X_{t_2}, X_{t_3} \middle| X_{t_1} = x_1\right] = \begin{bmatrix} \text{Cov}\,(X_{t_2}, X_{t_2}) & \text{Cov}\,(X_{t_2}, X_{t_3}) \\ \text{Cov}\,(X_{t_3}, X_{t_2}) & \text{Cov}\,(X_{t_3}, X_{t_3}) \end{bmatrix}$$
$$- \begin{bmatrix} \text{Cov}\,(X_{t_2}, X_{t_1}) \\ \text{Cov}\,(X_{t_3}, X_{t_1}) \end{bmatrix} \frac{1}{\text{E}X_{t_1}^2} \begin{bmatrix} \text{Cov}\,(X_{t_1}, X_{t_2}) \\ \text{Cov}\,(X_{t_1}, X_{t_3}) \end{bmatrix}^t$$
$$= \begin{bmatrix} t_2 & t_2 \\ t_2 & t_3 \end{bmatrix} - \begin{bmatrix} t_1 \\ t_1 \end{bmatrix} \frac{1}{t_1} \begin{bmatrix} t_1 & t_1 \end{bmatrix}$$
$$= \begin{bmatrix} t_2 - t_1 & t_2 - t_1 \\ t_2 - t_1 & t_3 - t_1 \end{bmatrix}$$
$$= \begin{bmatrix} s_1 & s_1 \\ s_1 & s_1 + s_2 \end{bmatrix}$$

The determinant of V is $s_1 s_2$ and with inverse $\frac{1}{s_1 s_2}\begin{bmatrix} s_1 + s_2 & -s_1 \\ -s_1 & s_1 \end{bmatrix}$.

$$u(x_2, x_3, s_2, s_3|x_1) = \frac{1}{2\pi s_1 s_2} \exp\left(-\frac{(s_1 + s_2)\, x_2^2 - 2 s_1 x_2 x_3 + s_1 x_3^2}{2 s_1 s_2}\right).$$

(b) Conditional on $X_{t_1} = x_1$ and $X_{t_3} = x_3$, find the distribution of X_{t_2}. This is $\mathcal{N}(\mu, \sigma^2)$ for some μ and σ^2 that depend on x_1, x_3, s_1, and s_2. Hint: The conditional density of X_{t_2} is the exponential of a quadratic. Identify the mean and variance by completing the square in the exponent.

Sol: Supposed we are looking for

$$u(X_{t_2} | X_{t_1} = x_1 \text{ and } X_{t_3} = x_3) \sim \mathcal{N}\left(\frac{t_3 - t_2}{t_3 - t_1}x_1 + \frac{t_2 - t_1}{t_3 - t_1}x_3, \frac{(t_3 - t_2)(t_2 - t_1)}{t_3 - t_1}\right).$$

Since

$$\begin{aligned}
\mathrm{E}\left[X_{t_2} | X_{t_1} = x_1 \text{ and } X_{t_3} = x_3\right] &= \mathrm{E}(X_{t_2}) + \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_1 & t_1 \\ t_1 & t_3 \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - \begin{bmatrix} \mathrm{E}(x_1) \\ \mathrm{E}(x_3) \end{bmatrix}\right) \\
&= \frac{1}{t_1(t_3 - t_1)} \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_3 & -t_1 \\ -t_1 & t_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\
&= \frac{1}{t_1(t_3 - t_1)} \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_3 x_1 - t_1 x_3 \\ -t_1 x_1 + t_1 x_3 \end{bmatrix} \\
&= \frac{t_3 - t_2}{t_3 - t_1}x_1 + \frac{t_2 - t_1}{t_3 - t_1}x_3. \\
&= \frac{s_2}{s_1 + s_2}x_1 + \frac{s_1}{s_1 + s_2}x_3.
\end{aligned}$$

Also, the variance is

$$\begin{aligned}
\mathrm{V}\left[X_{t_2} | X_{t_1} = x_1 \text{ and } X_{t_3} = x_3)\right] &= t_2 - \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_1 & t_1 \\ t_1 & t_3 \end{bmatrix}^{-1} \begin{bmatrix} t_1 & t_2 \end{bmatrix}^t \\
&= t_2 - \frac{t_3 t_1^2 - 2t_1^2 t_2 + t_1 t_2^2}{t_1(t_3 - t_1)} \\
&= \frac{t_2 t_3 - t_2 t_1 - t_3 t_1 + 2t_1 t_2 - t_2^2}{(t_3 - t_1)} \\
&= \frac{(t_2 - t_1)(t_3 - t_2)}{(t_3 - t_1)} \\
&= \frac{s_1 s_2}{s_1 + s_2}
\end{aligned}$$

Just do the below for fun,

$$\begin{aligned}
\Sigma^2(X_{t_1}, X_{t_2}) &= \begin{bmatrix} \mathrm{Cov}\,(X_{t_1}, X_{t_1}) & \mathrm{Cov}\,(X_{t_1}, X_{t_2}) \\ \mathrm{Cov}\,(X_{t_2}, X_{t_1}) & \mathrm{Cov}\,(X_{t_2}, X_{t_2}) \end{bmatrix} \\
&= \begin{bmatrix} t_1 & t_1 \\ t_1 & t_2 \end{bmatrix},
\end{aligned}$$

which has determinant $t_1(t_2 - t_1) = t_1 s_1$, and whose inverse is $\frac{1}{t_1 s_1}\begin{bmatrix} t_2 & -t_1 \\ -t_1 & t_1 \end{bmatrix}$.

Therefore, the joint density is

$$u(x_1, x_2, t_1, s_1) = \frac{1}{2\pi\sqrt{t_1 s_1}} \exp\left(-\frac{1}{2}\frac{t_2 x_1^2 - 2t_1 x_1 x_2 + t_1 x_2^2}{t_1 s_1}\right).$$

So the density

$$\begin{aligned} u(x_2, s_1 | \, x_1) &= \frac{u(x_1, x_2)}{\int_{\mathbb{R}} u(x_1, x_2) dx_2} \\ &= \frac{\frac{1}{2\pi\sqrt{t_1 s_1}} \exp\left(-\frac{1}{2}\frac{t_2 x_1^2 - 2t_1 x_1 x_2 + t_1 x_2^2}{t_1 s_1}\right)}{\frac{1}{2\pi\sqrt{t_1 s_1}} \exp\left(-\frac{1}{2}\frac{(t_2 - t_1) x_1^2}{t_1 s_1}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\frac{(x_2 - x_1)^2}{s_1}\right) dx_2} \\ &= \frac{\exp\left(-\frac{1}{2}\frac{(x_1 - x_2)^2}{s_1}\right)}{\sqrt{2\pi s_1}} \sim \mathcal{N}(x_1, s_1). \end{aligned}$$

Similarly,

$$\Sigma^2(X_{t_2}, X_{t_3}) = \begin{bmatrix} t_2 & t_2 \\ t_2 & t_3 \end{bmatrix},$$

which has determinant $t_2(t_3 - t_2) = t_2 s_2$, and whose inverse is $\frac{1}{t_2 s_2}\begin{bmatrix} t_3 & -t_2 \\ -t_2 & t_2 \end{bmatrix}$. So the conditional density

$$\begin{aligned} u(x_2, s_1 | \, x_3) &= \frac{u(x_2, x_3)}{\int_{\mathbb{R}} u(x_2, x_3) dx_2} \\ &= \frac{\frac{1}{2\pi\sqrt{t_2 s_2}} \exp\left(-\frac{1}{2}\frac{t_3 x_2^2 - 2t_2 x_2 x_3 + t_2 x_3^2}{t_2 s_2}\right)}{\frac{1}{2\pi\sqrt{t_2 s_2}} \exp\left(-\frac{1}{2}\frac{(t_3 - t_2) x_3^2}{t_3 s_2}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\frac{t_3 (x_2 - \frac{t_2}{t_3} x_3)^2}{t_2 s_2}\right) dx_2} \\ &= \frac{\exp\left(-\frac{1}{2}\frac{\left(x_2 - \frac{t_2}{t_3} x_3\right)^2}{t_2 s_2 / t_3}\right)}{\sqrt{2\pi t_2 s_2 / t_3}} \sim \mathcal{N}(\frac{t_2}{t_3} x_3, \frac{t_2}{t_3} s_2). \end{aligned}$$

(c) Show that the formula of part (b) is the same as the conditional density of X_{t_2} given any number of additional values for times $t_k < t_1$ and/or $t_k > t_3$. For example, conditioning on $X_{t_4} = x_4$ with $t_4 > t_3$ does not change the answer to (b) in the sense that x_4 and t_4 do not appear in the answer.

Sol: Consider $t_k < t_1$ first, without loss of generality, setting $t_k = t_0$

$$
\begin{aligned}
\mathrm{E}\left[X_{t_2} \middle| x_0, x_1, x_3\right] &= \begin{bmatrix} t_0 & t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_0 & t_0 & t_0 \\ t_0 & t_1 & t_1 \\ t_0 & t_1 & t_3 \end{bmatrix}^{-1} \left(\begin{bmatrix} x_0 \\ x_1 \\ x_3 \end{bmatrix} \right) \\
&= \frac{\begin{bmatrix} t_0 & t_1 & t_2 \end{bmatrix}}{t_0(t_1-t_0)(t_3-t_1)} \begin{bmatrix} t_1 t_3 - t_1^2 & t_0 t_1 - t_0 t_3 & 0 \\ t_1 t_0 - t_0 t_3 & t_0 t_3 - t_0^2 & t_0^2 - t_0 t_1 \\ 0 & t_0^2 - t_0 t_1 & t_0 t_1 - t_0^2 \end{bmatrix} \left(\begin{bmatrix} x_0 \\ x_1 \\ x_3 \end{bmatrix} \right) \\
&= \frac{(t_2 - t_1)x_3 + (t_3 - t_2)x_1}{(t_3 - t_1)} \\
&= \frac{t_3 - t_2}{t_3 - t_1} x_1 + \frac{t_2 - t_1}{t_3 - t_1} x_3. \\
&= \frac{s_2}{s_1 + s_2} x_1 + \frac{s_1}{s_1 + s_2} x_3.
\end{aligned}
$$

Also, the variance is

$$
\begin{aligned}
\mathrm{V}\left[X_{t_2} \middle| X_{t_1} = x_1 \text{ and } X_{t_3} = x_3)\right] &= t_2 - \begin{bmatrix} t_0 & t_1 & t_2 \end{bmatrix} \begin{bmatrix} t_0 & t_0 & t_0 \\ t_0 & t_1 & t_1 \\ t_0 & t_1 & t_3 \end{bmatrix}^{-1} \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix} \\
&= t_2 - \frac{t_3 t_1^2 - 2t_1^2 t_2 + t_1 t_2^2}{t_1(t_3 - t_1)} \\
&= \frac{t_2 t_3 - t_2 t_1 - t_3 t_1 + 2t_1 t_2 - t_2^2}{(t_3 - t_1)} \\
&= \frac{(t_2 - t_1)(t_3 - t_2)}{(t_3 - t_1)} \\
&= \frac{s_1 s_2}{s_1 + s_2}
\end{aligned}
$$

Similarly, one can obtain t_4 not involved.

(d) Specialize to the case $s_1 = s_2 = \Delta t$. Compare the variance of X_{t_2} with both X_{t_1} and X_{t_3} specified to the variance with only X_{t_1} specified.

(e) (*not an action item*) You can use these formulas to generate Brownian motion paths in a different way. First generate $X_1 \sim \mathcal{N}(0,1)$ and $X_0 = 0$. Then use the result of (d) to generate $X_{1/2} \sim \mathcal{N}(\cdot,\cdot)$ (results of (d)). Then use (d) again to generate $X_{1/4}$ using X_0 and $X_{1/2}$, and $X_{3/4}$ from $X_{1/2}$ and X_1. Continuing in this way you can make a Brownian motion path in as much detail as you want.

4. (*backward equation*) Let X_t be a standard Brownian motion starting from $X_0 = 0$. Let

$$\tau = \min\left\{t \ \text{ so that } \ |X_t| = 1\right\}.$$

Find the expected hitting time $\mathrm{E}[\tau]$. Hint:

(a) Suppose $V(x,t)$ is a running time reward function and the total reward starting from x at time t is

$$\int_t^\tau V(X_s,s)\,ds\,.$$

There the process starts with $X_t = s$, and τ is the first hitting time after t, and $|x| \leq 1$. Define the value function for this to be

$$f(x,t) = \mathrm{E}_{x,t}\left[\int_t^\tau V(X_s,s)\,ds\right]\,.$$

Figure out the PDE that f satisfies.

Sol: We write $X_{t+\triangle t} = X_t + \triangle x$ and expand $f(X_{t+\triangle t}, t+\triangle t)$ in a taylor series.

$$\begin{aligned} f(X_{t+\triangle t}, t+\triangle t) = f(X_t,t) \\ &+ \partial_x f(X_t,t)\triangle X \\ &+ \partial_t f(X_t,t)\triangle t \\ &+ \frac{1}{2}\partial_x^2 f(X_t,t)\triangle X^2 \\ &+ \mathcal{O}(|\triangle x|^3) + \mathcal{O}(|\triangle X|\,|\triangle t|) + \mathcal{O}(\triangle t^2). \end{aligned}$$

The three remainder terms on the last line are the sizes of the three lowest order Taylor series terms left out. Considerting the tower property, whcih says that the algebra $\mathcal{F}_{t+\triangle t}$ has a little more information than $\mathcal{F}_t$. Therefore, if Y is any random variable, we must have

$$\mathrm{E}\left[\mathrm{E}\left[Y|\,\mathcal{F}_{t+\triangle t}\right]|\,\mathcal{F}_t\right] = \mathrm{E}\left[Y|\,\mathcal{F}_t\right].$$

Thus,

$$\begin{aligned} f(x,t) &= \mathrm{E}_{x,t}\left[\int_t^\tau V(X_s,s)\,ds\right] \\ &= \mathrm{E}\left[\int_t^\tau V(X_s,s)\,ds\,\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\int_t^\tau V(X_s,s)\,ds\,\middle|\,\mathcal{F}_{t+\triangle t}\right]\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{E}\left[f(X_{t+\triangle t}, t+\triangle t)|\,\mathcal{F}_t\right] \end{aligned}$$

Now take the expectation of both sides conditioning on $\mathcal{F}_t$ and pull out of the expectation anything that is known in $\mathcal{F}_t$:

$$\begin{aligned}
\mathrm{E}\left[f(X_{t+\triangle t}, t+\triangle t) \middle| \mathcal{F}_t\right] = f(x,t) \\
&+ \partial_x f(x,t)\mathrm{E}\left[\triangle X \middle| \mathcal{F}_t\right] \\
&+ \partial_t f(x,t)\triangle t \\
&+ \frac{1}{2}\partial_x^2 f(x,t)\mathrm{E}\left[\triangle X^2 \middle| \mathcal{F}_t\right] \\
&+ \mathcal{O}(\mathrm{E}\left[|\triangle X|^3 \middle| \mathcal{F}_t\right]) + \mathcal{O}(\mathrm{E}\left[|\triangle X|^3 \middle| \mathcal{F}_t\right]\triangle t) + \mathcal{O}(\triangle t^2)
\end{aligned}$$

$$\begin{aligned}
f(x,t) &= \mathrm{E}\left[\int_t^{t+\triangle t} V(X_s,s)ds + \int_{t+\triangle t}^{\tau} V(X_s,s)\,ds \middle| \mathcal{F}_t\right] \\
&= V(x,t)\triangle t + \mathrm{E}\left[f(X_{t+\triangle t}, t+\triangle t) \middle| \mathcal{F}_t\right] \\
&= f(x,t) + \left(V(x,t) + \partial_t f(x,t) + \frac{1}{2}\partial_x^2 f(x,t)\right)\triangle t + \mathcal{O}\left(\triangle t^{\frac{1}{2}}\right)
\end{aligned}$$

Taking $\triangle t \to 0$ shows that f satisfies the backward equation. The Dirichlet boundary condition is $f(1,t) = 0$ and $f(-1,t) = 0$.

(b) The case $V(x,t) = 1$ gives the expected hitting time.

Sol: Considering the partial differential equation

$$\begin{cases} 1 + f_t + \frac{1}{2}f_{xx} = 0 & -1 < x < 1 \\ f(1,t) = f(-1,t) = 0 \end{cases}$$

Supposed that $f(x,t) = X(x)T(t)$, then

$$\frac{T'}{T} = -\frac{1}{2}\frac{X''}{X} = \lambda < 0.$$

So $T(t) = e^{\lambda t}$, and

$$\begin{aligned}
X(x) &= A\cos(\sqrt{2\lambda}x) + B\sin(\sqrt{2\lambda}x), \\
X(-1) &= 0 = X(1).
\end{aligned}$$

Solving this we get $A = 0 = B$. Thus,

$$\begin{aligned}
f(x,t) &= e^{\lambda t}\cdot 0 - x^2 + 1 \\
&= 1 - x^2.
\end{aligned}$$

Setting $V(x,t) = 1$ we have

$$\begin{aligned}
\mathrm{E}_{x,t}\left[\tau - t\right] &= f(x,t) \\
&= 1 - x^2
\end{aligned}$$

Therefore,

$$\mathrm{E}_{x,t}\left[\tau\right] = 1 - x^2 + t.$$

Differentiation of $\mathrm{E}\left[e^{-\alpha\tau}\right] = e^{-\sqrt{2\alpha}}$ with respect to α results in

$$\begin{aligned}
\mathrm{E}\left[\tau\right] &= -\partial_\alpha \mathrm{E}\left[e^{-\alpha\tau}\right]\Big|_{\alpha=0} \\
&= -\partial_\alpha \left(e^{-\sqrt{2\alpha}}\right)\Big|_{\alpha=0} \\
&= \frac{1}{2e^{\sqrt{2\alpha}}\sqrt{2\alpha}}\Bigg|_{\alpha=0} \\
&= \partial_\alpha \left(-1 + \sqrt{2\alpha} - \frac{1}{2}2\alpha + \mathcal{O}(\alpha^{\frac{3}{2}})\right)\Bigg|_{\alpha=0} \\
&= \infty
\end{aligned}$$

for all $\alpha > 0$.

(c) There is a subtlety here that we need to show $\mathrm{E}[\,\tau\,] < \infty$. The assignment for a future week will show that there is a $x > 0$ so that $\mathrm{P}_{x,0}(\tau > t) \le e^{-ct}$.

Sol:

5. (*Computing*) **New this week**: *Download the file* ***`coding.pdf`*** *. It contains guidelines for coding. Ultimately they will save you time in the computing assignments.* The material for this week contains the PDF

$$M_t = \max_{0 \le s \le t} X_s$$

and a formula for

$$S_{t,a}(x)dx = \mathrm{P}(x \le X_t \le x + dx \mid X_s < a \text{ for } 0 \le s \le t)$$

You made histograms of these distributions last week. This week, put the exact formulas on the graphs to see whether they agree. Play with parameters to see how good a fit you can get in a reasonable amount of computer time.