Stochastic Calculus, Courant Institute, Fall 2012
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2012/index.html
Always check the class message board on the blackboard site from home.nyu.edu before doing any work on the assignment.

Assignment 5, due October 29

Corrections: (none yet.)

1. (*A generalization of the Ito isometry formula*) Here is a handy to calculate some things about Ito integrals

   (a) Suppose f_t and g_t are non-anticipating functions, and the corresponding Ito integrals are

$$X_t = \int_0^t f_s dW_s$$
$$Y_t = \int_0^t g_s dW_s .$$

Show that

$$\text{cov}(X_t, Y_t) = \text{E}[\, X_t Y_t] = \int_0^t \text{E}[\, f_s g_s]\, ds \; .$$

Sol: Recall, a non-anticipating process (or adapted process) is one that cannot see into the future. An informal interpretation is that X is adapted if and only if, for every realization and every n, X_n is known at time n. The concept of an adapted process is essential, for instance, in the definition of the Ito integral, which only makes sense if the integrand is an adapted process. Consider the *white noise* representation of Brownian motion. White noise is a generalized function (distribution). Brownian motion can be thought of as the motion of a particle pushed by white noise, that it $W_t = \int_0^t \xi(s)\, ds$. Recall that $\xi(t)$ has mean zero and

$$\text{E}\left[\xi(t_1)\,\xi(t_2)\right] = \delta(t_1 - t_2) \, .$$

Then $dW_t = \xi(t)dt$

$$\begin{aligned}
\operatorname{cov}(X_t, Y_t) &= \mathrm{E}\left[\int_0^t f_{s_1} dW_{s_1} \int_0^t g_{s_2} dW_{s_2}\right] \\
&= \int_0^t \int_0^t \mathrm{E}\left[f_{s_1} g_{s_2} \xi(s_1)\xi(s_2)\right] ds_1 ds_2 \\
&= \int_0^t \int_0^t \mathrm{E}\left[f_{s_1} g_{s_2}\right] \mathrm{E}\left[\xi(s_1)\xi(s_2)\right] ds_1 ds_2 \\
&= \int_0^t \int_0^t \mathrm{E}\left[f_{s_1} g_{s_2}\right] \delta(s_1 - s_2) ds_1 ds_2 \\
&= \int_0^t \mathrm{E}\left[f_s g_s\right] ds
\end{aligned}$$

(b) Suppose $f_t = t^2$ and $g_t = 1$. The notes for Week 5 show that $X_t \sim \mathcal{N}(0, t^5/5)$. Clearly $Y_t = W_t$. Compute the covariance of X_t and W_t using the result of part (a). This should agree with the result of question (3) from Assignment 3.

Sol: To compute the covariance of X_t and W_t, $\mathrm{E}\left[f_s g_s\right] = \mathrm{E}\left[s^2\right] =$

$$\begin{aligned}
\operatorname{cov}(X_t, W_t) &= \int_0^t \mathrm{E}\left[f_s g_s\right] ds \\
&= \int_0^t s^2 ds \\
&= \frac{t^3}{3}.
\end{aligned}$$

(c) Since (X_t, W_t) is a bivariate normal whose variance/covariance structure you know, you can compute the conditional variance $\operatorname{var}(X_t|W_t)$. Use this result to show that X_t is not a function of W_t. This is an example of a general phenomenon, that the value of an Ito integral depends on the whole path $W_{[0,t]}$, not just the endpoint W_t.

Sol: The correlation is just $\rho = \frac{\operatorname{cov}(X_t, W_t)}{\sigma_{X_t} \sigma_{W_t}} = \frac{t^3/3}{(t^5/5)^{1/2} t^{1/2}} = \sqrt{5}/3$ a constant. Thus,

$$\begin{aligned}
\operatorname{var}(X_t|W_t) &= \sigma_X^2 \left(1 - \rho^2\right) \\
&= \frac{4}{9}\frac{t^5}{5} = \frac{4}{45} t^5.
\end{aligned}$$

Note that $X_t = \int_0^t s^2 dW_s$ depends on the whole path W_s for $s \in [0, t]$, not just the endpoint W_t.

2. (*Ornstein Uhlenbeck*) This exercise goes through another approach to the Ornstein Uhlenbeck process. This time the process is called V_t because it represents the velocity of a small particle in a fluid at time t. This particle is subject to a random force F_t and friction with coefficient γ. We assume the units have been chosen so the mass of the particle is 1. The dynamics are

$$\frac{dV_t}{dt} = -\gamma V_t + F_t \ . \tag{1}$$

The term $-\gamma V_t$ represents friction proportional to the velocity of the particle, but pushing in the direction the particle is not moving. We always assume $\gamma > 0$.

(a) Write the solution that corresponds to $F \equiv 0$ and $V_s = 1$. Call this $G(s,t)$. This plays the role that is played by a Green's function for a PDE.

Sol: It's obvious that the solution of (1) is given by

$$V_t = Ce^{-\gamma t},$$

Substitute the boundary condition $V_s = 1$, we get $V_s = 1 = Ce^{-\gamma s}$, in which we know $C = e^{\gamma s}$. Thus

$$V_t = e^{-\gamma(t-s)} := G\left(s,t\right).$$

(b) Suppose F_t is a bounded function of t or grown slowly with t as $t \to -\infty$. Suppose that V_t likewise grows slowly with t or is bounded. Show that

$$V_t = \int_{-\infty}^{t} G(s,t) F_s \, ds \ . \tag{2}$$

Hint: Differentiate with respect to t. There are two terms, which correspond to the two terms on the right of (1).

Sol: Differentiate with respect to t on both side of equation (2),

$$\begin{aligned}\frac{dV_t}{dt} &= -\gamma \int_{-\infty}^{t} G\left(s,t\right) F_s \, ds + G\left(t,t\right) F_t \\ &= -\gamma \int_{-\infty}^{t} G\left(s,t\right) F_s \, ds + F_t \\ &= -\gamma V_t + F_t\end{aligned}$$

from (1). Thus $V(t) = \int_{-\infty}^{t} G\left(s,t\right) F_s \, ds$ as desired.

(c) An *impulsive* force of size 1 has the form $F_t = \delta(t - t_0)$. Describe the solution (2) with an impulsive force, both for $t < t_0$ and $t > t_0$.

Sol: For $t < t_0$,

$$V_t = \int_{-\infty}^{t} G(s,t)\delta(s-t_0)\,ds = 0,$$

and for $t > t_0$

$$\begin{aligned} V_t &= \int_{-\infty}^{t} G(s,t)\delta(s-t_0)\,ds \\ &= G(t_0,t) \\ &= e^{-\gamma(t-t_0)}. \end{aligned}$$

(d) Show that the formula (2) corresponds to a superposition of impulsive forces at time s over the interval ds with size $F_s ds$.

Sol: The impulsive forces at time s over the interval ds with size $F_s ds$ is given by

$$V_t = \int_{-\infty}^{t} G(s,t)\delta(ds)\,F_s ds.$$

(e) Suppose we replace the impulsive force over the interval $(s, s+ds)$ with a mean zero Gaussian σdW_s in (2). Find a formula for $\mathrm{E}[\,V_t]$ and one for $\mathrm{var}(V_t)$. Hint: These are independent of t, why? If you are uncomfortable with an infinite range of integration, you may replace $-\infty$ by a large negative t_0 in (2), then let $t_0 \to -\infty$.

Sol: Replacing the impulsive forces by σdW_s, which gives us

$$\begin{aligned} V_t &= \lim_{t_0 \to -\infty} \int_{t_0}^{t} G(s,t)\sigma dW_s \\ &= \lim_{t_0 \to -\infty} \left(\underbrace{e^{-\gamma(t-t_0)} V_{t_0}}_{\text{neglegible}} + \int_{t_0}^{t} G(s,t)\sigma dW_s \right) \\ &= \int_{-\infty}^{t} G(s,t)\sigma dW_s \end{aligned}$$

Thus $\mathbb{E}\left[V_t\right] = 0$. Another way to see this is by considering the method of integrating factor

$$
\begin{aligned}
d\left(e^{\gamma t}V_t\right) &= e^{\gamma t}\sigma dW_t \\
\Rightarrow e^{\gamma t}V_t - e^{\gamma t_0}V_{t_0} &= \int_{-\infty}^{t} e^{\gamma s}\sigma dW_s \\
\Rightarrow V_t &= e^{-\gamma(t-t_0)}V_{t_0} + \int_{-\infty}^{t} e^{-\gamma(t-s)}\sigma dW_s \\
&= \int_{-\infty}^{t} e^{-\gamma(t-s)}\sigma dW_s .
\end{aligned}
$$

Let us calculate the variance,

$$
\begin{aligned}
\mathrm{Var}\left[V_t\right] &= \mathrm{E}\left[V_t^2\right] - \left(\mathrm{E}\left[V_t\right]\right)^2 \\
&= \sigma^2 e^{-2\gamma t}\int_{-\infty}^{t} e^{2\gamma s}ds \\
&= e^{-2\gamma t}\frac{\sigma^2}{2\gamma}\left(e^{2\gamma t}-0\right) = \frac{\sigma^2}{2\gamma}.
\end{aligned}
$$

Notice that these are independent of t, since we integrating from $-\infty$.

(f) Show that the V_t given by part (2e) is a Markov process. Hint: write a formula for $\mathrm{E}[\,V_s \mid \mathcal{F}_t]$ that depends only on V_t and $W_{[t,s]}$.

Sol: The V_t given by part (2e) is $V_t = \int_{-\infty}^{t} G(s,t)\sigma dW_s$. So for $s > t$, we must have

$$
\mathbb{E}\left[\int_t^s G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] = 0
$$

by the martingale property of Ito integrals. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[V_s\middle|\mathcal{F}_t\right] &= \mathbb{E}\left[\int_{-\infty}^{s} G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{t} G(s',s)\sigma dW_{s'} + \int_t^s G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{t} G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] + \mathbb{E}\left[+\int_t^s G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{t} G(s',s)\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= e^{-\gamma(s-t)}\mathbb{E}\left[\int_{-\infty}^{t} e^{-\gamma(t-s')}\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= G(t,s)\mathbb{E}\left[\int_{-\infty}^{t} G(s',t)\sigma dW_{s'}\middle|\mathcal{F}_t\right] \\
&= G(t,s)\mathbb{E}\left[V_t\middle|\mathcal{F}_t\right] \\
&= G(t,s)V_t .
\end{aligned}
$$

which depends only on V_t at time t, but not depends on the path that V followed to get here. Thus $\mathbb{E}\left[V_s|\,\mathcal{F}_t\right] = \mathbb{E}\left[V_s|\,V_t\right]$, which is the Markov property.

(g) Suppose $\Delta V = V_{t+\Delta t} - V_t$. Find a formula for $\mathrm{E}[\,\Delta V \mid \mathcal{F}_t]$, and one for $\mathrm{E}\Big[\,(\Delta V)^2 \mid \mathcal{F}_t\Big]$. Show that this is the same as the Ornstein Uhlenbeck process in that the formulas here agree with the formulas from Week 3, Section 5 (possibly with 2γ for γ or 2σ for σ. Hint: for the latter, it may be simpler to calculate $\mathrm{var}\,(\Delta V \mid \mathcal{F}_t)$, because the dependence on V_t is different.

Sol: Let us consider the process ΔV first,

$$
\begin{aligned}
\Delta V &= V_{t+\Delta t} - V_t \\
&= \int_{-\infty}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s - \int_{-\infty}^{t} e^{-\gamma(t-s)}\sigma dW_s \\
&= \int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s + \int_{-\infty}^{t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s - \int_{-\infty}^{t} e^{-\gamma(t-s)}\sigma dW_s \\
&= \int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s + (e^{-\gamma\Delta t} - 1)\int_{-\infty}^{t} e^{-\gamma(t-s)}\sigma dW_s \\
&= \int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s + (e^{-\gamma\Delta t} - 1)V_t.
\end{aligned}
$$

Thus the expected value is

$$
\begin{aligned}
\mathbb{E}\left[\Delta V \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s + (e^{-\gamma\Delta t} - 1)V_t \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s \mid \mathcal{F}_t\right] + (e^{-\gamma\Delta t} - 1)V_t \\
&= 0 + (e^{-\gamma\Delta t} - 1)V_t \\
&= -\gamma V_t \Delta t + \mathcal{O}(\Delta t^2),
\end{aligned}
$$

by Taylor's expansion $e^x - 1 = \sum_{n=1}^{\infty} x^n/n!$. For the second moment,

$$
\begin{aligned}
\mathbb{E}\left[(\Delta V)^2 \mid \mathcal{F}_t\right] &= \mathbb{E}\left[V_{t+\Delta t}^2 - 2V_tV_{t+\Delta t} + V_t^2 \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[V_{t+\Delta t}^2 \mid \mathcal{F}_t\right] - 2V_t\mathbb{E}\left[V_{t+\Delta t} \mid \mathcal{F}_t\right] + V_t^2 \\
&= \mathbb{E}\left[V_{t+\Delta t}^2 \mid \mathcal{F}_t\right] - 2e^{-\gamma\Delta t}V_t^2 + V_t^2.
\end{aligned}
$$

Let us consider the first term of the above integral,

$$\begin{aligned}
\mathbb{E}\left[V_{t+\Delta t}^2 \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\left(\int_{-\infty}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s\right)^2 \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\left(\int_{-\infty}^{t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s\right)^2 \mid \mathcal{F}_t\right] \\
&+ 2\mathbb{E}\left[\left(\int_{-\infty}^{t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s\right) \mid \mathcal{F}_t\right] \mathbb{E}\left[\left(\int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s\right) \mid \mathcal{F}_t\right] \\
&+ \mathbb{E}\left[\left(\int_{t}^{t+\Delta t} e^{-\gamma(t+\Delta t-s)}\sigma dW_s\right)^2 \mid \mathcal{F}_t\right] \\
&= e^{-2\gamma\Delta t}V_t^2 + 2e^{-\gamma\Delta t}V_t \cdot 0 + \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma\Delta t}\right).
\end{aligned}$$

Combining above we get

$$\begin{aligned}
\mathbb{E}\left[(\Delta V)^2 \mid \mathcal{F}_t\right] &= e^{-2\gamma\Delta t}V_t^2 + \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma\Delta t}\right) - 2e^{-\gamma\Delta t}V_t^2 + V_t^2 \\
&= V_t^2(1 + e^{-2\gamma\Delta t} - 2e^{-\gamma\Delta t}) + \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma\Delta t}\right) \\
&= V_t^2(1 + 1 + -2\gamma\Delta t - 2 + 2\gamma\Delta t) + \frac{\sigma^2}{2\gamma}(1 - 1 + 2\gamma\Delta t) + \mathcal{O}(\Delta t^2) \\
&= \sigma^2\Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

3. In Einstein's model of Brownian motion, the location of a particle is

$$X_t = \int_0^t V_s \, ds \, . \tag{3}$$

This exercise shows that this is true, provided we use an appropriate scaling. The parameter γ from Exercise (2) controls how fast V_t loses memory. Therefore, in this exercise we take the limit $\gamma \to \infty$ and identify the limiting process X_t.

(a) Find a formula for X_t in the form

$$X_t = \int_0^t L(s,t) dW_s.$$

Hint: Combine the two formulas (2) (with $F_s ds = dW_s$) and (3), reverse the order of integration.

Sol: Substitute (2), then $V_{s'} = \int_{-\infty}^{s'} G(s,s')\sigma dW_s$

$$
\begin{aligned}
X_t &= \int_0^t \int_{-\infty}^{s'} G(s,s')\sigma dW_s ds' \\
&= \int_0^t \left(\int_{-\infty}^{0} G(s,s')\sigma dW_s + \int_0^s G(s,s')\sigma dW_s \right) ds' \\
&= \int_0^t \int_{-\infty}^{0} G(s,s')\sigma dW_s ds' + \int_0^t \int_0^s G(s,s')\sigma dW_s ds' \\
&= \int_0^t e^{-\gamma s'} V_0 ds' + \sigma \int_0^t \int_s^t G(s,s') ds' dW_s \\
&= \left(\frac{1-e^{-\gamma t}}{\gamma} \right) V_0 + \int_0^t \sigma \left(\frac{1-e^{-\gamma(t-s)}}{\gamma} \right) dW_s.
\end{aligned}
$$

Since the first term goes to 0 as $\gamma \to \infty$, we might drop it. So we can write X_t as $X_t = \int_0^t L(s,t) dW_s$ where

$$
L(s,t) = \sigma \left(\frac{1-e^{-\gamma(t-s)}}{\gamma} \right).
$$

(b) Find a formula for σ as $\gamma \to \infty$ so that $\mathrm{E}\left[X_1^2 \right] = 1$. Call this process $X_{\gamma,t}$ Hint: the exact formula for finite γ may be hard to find, but you can find the behavior of σ as $\gamma \to \infty$ as a power of γ to leading order. This all you need.

Sol: Apply Ito isometry,

$$
\begin{aligned}
\mathbb{E}\left[X_1^2 \right] &= \mathbb{E}\left[\int_0^1 L^2(s,1) ds \right] \\
&= \mathbb{E}\left[\sigma^2 \int_0^1 \left(\frac{1-e^{-\gamma(1-s)}}{\gamma} \right)^2 ds \right] \\
&= \mathbb{E}\left[\sigma^2 \int_0^1 \left(\frac{1-2e^{-\gamma(1-s)} + e^{-2\gamma(1-s)}}{\gamma^2} \right) ds \right] \\
&= \frac{\sigma^2}{\gamma^2} - \frac{2\sigma^2}{\gamma^3}(1-e^{-\gamma}) + \frac{\sigma^2}{2\gamma^3}(1-e^{-2\gamma}) \\
&\to 1.
\end{aligned}
$$

Therefore, $\sigma = \gamma$. Let us call the process

$$
\begin{aligned}
X_{\gamma,t} &= \int_0^t L(s,t) dW_s \\
&= \int_0^t \left(1 - e^{-\gamma(t-s)} \right) dW_s.
\end{aligned}
$$

(c) Show that in the limit $\gamma \to \infty$ from part (3b) the process $X_{\gamma,t}$ has $X_{\gamma,[0,T]} \overset{\mathcal{D}}{\to} X_t$ as $\gamma \to \infty$, where X_t is standard Brownian motion. Take this to mean that the finite dimensional joint distributions of $(X_{t_1}, \ldots, X_{t_n})$ are what they should be for Brownian motion. Hint: Since $X_{\gamma,[0,T]}$ is Gaussian (being a linear function of $W_{[0,T]}$), you just need to evaluate the limiting means and covariances. You can do these from part (3b) and the independent increments property. So you need to show that as $\gamma \to \infty$, you approach independent increments.

Sol: Intuitively, in the limit $\gamma \to \infty$ from part (3b) the process

$$\begin{aligned} dX_{\gamma,t} &= \left(1 - e^{-\gamma(t-s)}\right) dW_s \Big|_0^T \\ &\to dW_t \end{aligned}$$

has $X_{\gamma,[0,T]} \overset{\mathcal{D}}{\to} X_t$ as $\gamma \to \infty$, where X_t is standard Brownian motion. We show this fact by considering the finite dimensional joint distributions of $(X_{t_1}, \ldots, X_{t_n})$ are what they should be for Brownian motion. Following from the hint, since $X_{\gamma,[0,T]}$ is Gaussian, we first evaluate the limiting mean, $X_{\gamma,t}$ that has mean 0 and variance t. Notice that if we write $X_{\gamma,t}$ either to be

$$X_{\gamma,t} = \frac{1 - e^{-\gamma t}}{\gamma} V_0 + \int_0^t 1 - e^{-\gamma(t-s)} dW_s$$

or

$$X_{\gamma,t} = \int_0^t 1 - e^{-\gamma(t-s)} dW_s,$$

we both have $\mathbb{E}\left[X_{\gamma,t}\right] = 0$ and $\operatorname{var}\left[X_{\gamma,t}\right] = t$ as $\gamma \to \infty$ and $\sigma = \gamma$. To show that $X_{\gamma,t}$ has independent increment as $\gamma \to \infty$, we consider the non-overlapping intervals $t_1 < t_2 < t_3 < t_4$, then

$$\begin{aligned} X_{\gamma,t_2} - X_{\gamma,t_1} &= \int_0^{t_2} 1 - e^{-\gamma(t_2-s)} dW_s - \int_0^{t_1} 1 - e^{-\gamma(t_1-s)} dW_s \\ &= \int_{t_1}^{t_2} 1 - e^{-\gamma(t_2-s)} dW_s + \int_0^{t_1} 1 - e^{-\gamma(t_2-s)} dW_s - \int_0^{t_1} 1 - e^{-\gamma(t_1-s)} dW_s \\ &= \int_{t_1}^{t_2} 1 - e^{-\gamma(t_2-s)} dW_s + \int_0^{t_1} e^{-\gamma(t_1-s)} - e^{-\gamma(t_2-s)} dW_s \\ X_{\gamma,t_4} - X_{\gamma,t_3} &= \int_0^{t_4} 1 - e^{-\gamma(t_4-s)} dW_s - \int_0^{t_3} 1 - e^{-\gamma(t_3-s)} dW_s \\ &= \int_{t_3}^{t_4} 1 - e^{-\gamma(t_4-s)} dW_s + \int_0^{t_3} e^{-\gamma(t_3-s)} - e^{-\gamma(t_4-s)} dW_s. \end{aligned}$$

So we are ready to examine the independent increment property,

$$
\begin{aligned}
&\mathbb{E}[(X_{\gamma,t_2} - X_{\gamma,t_1})(X_{\gamma,t_4} - X_{\gamma,t_3})] \\
&= \mathbb{E}[\left(\int_{t_1}^{t_2} 1 - e^{-\gamma(t_2-s)} dW_s\right)\left(\int_0^{t_3} e^{-\gamma(t_3-s)} - e^{-\gamma(t_4-s)} dW_s\right) \\
&+ \left(\int_0^{t_1} e^{-\gamma(t_1-s)} - e^{-\gamma(t_2-s)} dW_s\right)\left(\int_0^{t_3} e^{-\gamma(t_3-s)} - e^{-\gamma(t_4-s)} dW_s\right)] \\
&= \mathbb{E}[\int_{t_1}^{t_2}\int_{t_1}^{t_2} \left(1 - e^{-\gamma(t_2-s)}\right)\left(e^{-\gamma(t_3-s')} - e^{-\gamma(t_4-s')}\right) dW_s dW_{s'} \\
&+ \int_0^{t_1}\int_0^{t_1} \left(e^{-\gamma(t_1-s)} - e^{-\gamma(t_2-s)}\right)\left(e^{-\gamma(t_3-s')} - e^{-\gamma(t_4-s')}\right) dW_s dW_{s'}] \\
&= \int_{t_1}^{t_2} \left(1 - e^{-\gamma(t_2-s)}\right)\left(e^{-\gamma(t_3-s)} - e^{-\gamma(t_4-s)}\right) ds \\
&+ \int_0^{t_1} \left(e^{-\gamma(t_1-s)} - e^{-\gamma(t_2-s)}\right)\left(e^{-\gamma(t_3-s)} - e^{-\gamma(t_4-s)}\right) ds \\
&\to 0
\end{aligned}
$$

as $\gamma \to \infty$. Recall *Lévy's theorem*: let X_t be a stochastic process with continuous trajectories. Assume that it starts from zero, $X_0 = 0$ and also assume that X_t and $X_t^2 - t$ are martingales with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$. Then X_t is a Brownian motion starting from zero. *Lévy's theorem* confirms that X_t is standard Brownian motion.

(d) (*only for those who have taken Probability Limit Theorems II or otherwise have the background to understand the question*) Complete part (3c) by showing that the $X_{\gamma,[0,T]}$ form a tight family. You can do this by finding uniform estimates of the form

$$\mathrm{E}\left[\, \Delta X_\gamma^4 \right] \leq C \Delta t^2 \;,$$

which imply that the paths X_γ are uniformly Hölder continuous.

Sol: Recall that Kolmogorov theorem states that a family of probability measures is tight if there exist $\alpha > 0$, $\beta > 0$, $B < \infty$, and $C < \infty$ such that for all γ,

$$\mathbb{E} X_0^\beta \leq B$$

and

$$\mathbb{E}[\Delta X_\gamma^\beta] \leq C \Delta t^{1+\alpha}.$$

It's clear that for $\gamma > 0$ large enough,

$$
\begin{aligned}
1 - e^{-\gamma(t+\Delta t - s)} &= \gamma(t + \Delta t - s) + \mathcal{O}(\Delta t^2) \\
e^{-\gamma(t-s)} - e^{-\gamma(t+\Delta t - s)} &= 1 - \gamma(t-s) - 1 + \gamma(t + \Delta t - s) + \mathcal{O}(\Delta t^2) \\
&= \gamma \Delta t + \mathcal{O}(\Delta t^2)
\end{aligned}
$$

and therefore we shall have

$$
\begin{aligned}
\mathbb{E}[\Delta X_\gamma^4] &= \mathbb{E}\left(\int_t^{t+\Delta t} 1 - e^{-\gamma(t+\Delta t - s)} dW_s + \int_0^t e^{-\gamma(t-s)} - e^{-\gamma(t+\Delta t-s)} dW_s\right)^4 \\
&= \mathbb{E}[\underbrace{\int_t^{t+\Delta t} \gamma(t+\Delta t - s) dW_s}_{:=I_1} + \underbrace{\int_0^t \gamma \Delta t dW_s}_{:=I_2} + \mathcal{O}(\Delta t^2)]^4 \\
&= \mathbb{E}I_1^2\mathbb{E}I_1^2 + 6\mathbb{E}I_1^2\mathbb{E}I_2^2 + \mathbb{E}I_2^2\mathbb{E}I_2^2.
\end{aligned}
$$

Now let's compute $\mathbb{E}I_1^2$ and $\mathbb{E}I_2^2$:

$$
\begin{aligned}
\mathbb{E}I_1^2 &= \gamma^2 \int_t^{t+\Delta t} (t+\Delta t - s)^2 ds \\
&= \mathcal{O}(\Delta t) \\
\mathbb{E}I_2^2 &= \gamma^2 t \Delta t = \mathcal{O}(\Delta t).
\end{aligned}
$$

Thus $\mathbb{E}[\Delta X_\gamma^4] = \mathcal{O}(\Delta t^2)$ or equivalent saying that $\mathbb{E}[\Delta X_\gamma^4] \leq C\Delta t^2$. Here the case is simply $\alpha = 1$ and $\beta = 4$.

4. (*Strong law of large numbers*) Suppose Y_k is a family of i.i.d. random variables with $\mathrm{E}[\,Y_k] = \mu$. The *Kolmogorov strong law* of large numbers is the theorem that if $\mathrm{E}[\,|Y_k|] < \infty$, then

$$
\overline{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k \to \mu \ \text{ as } \ n \to \infty \ \ a.s. \tag{4}
$$

This exercise does not suggest his brilliant proof using the *three series lemma* or the more recent proof using the *Birkoff ergodic theorem*. Instead: Give a proof of (4) using the hypothesis $\mathrm{E}\left[\,Y_k^4\right] < \infty$. Hint: Suppose $\mu = 0$. The statement $\overline{Y}_n \to 0$ is the same as the statement $\overline{Y}_n^4 \to 0$. Set $X_n = \overline{Y}_n^4$ and try to show that $\sum \mathrm{E}[\,X_n] < \infty$ and use the Borel Cantelli style lemma from the notes. What do you do if $\mu \neq 0$?

Proof: First we suppose that $\mu = 0$, let $S_n = \sum_{k=1}^n Y_k$, then

$$
\begin{aligned}
\mathrm{E}\left[|S_n|^4\right] &= n\mathrm{E}\left[Y_1^4\right] + 3n\,(n-1)\left(EY_1^2\right)^2 \\
&\leq nC + 3n^2\sigma^4.
\end{aligned}
$$

Therefore for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}\left[\overline{Y}_n \geq \delta\right] &= \mathbb{P}\left[\left|\frac{S_n}{n}\right| \geq \delta\right] \\ &= \mathbb{P}\left[|S_n| \geq n\delta\right] \\ &\leq \frac{\mathrm{E}\left[|S_n|^4\right]}{n^4\delta^4} \\ &\leq \frac{nC + 3n^2\sigma^4}{n^4\delta^4},\end{aligned}$$

which is summable. Now since

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\left|\frac{S_n}{n}\right| \geq \delta\right] < \infty,$$

we can apply the Borel-Cantelli lemma and hence $\overline{Y}_n \to 0$ a.s..
If $\mu \neq 0$, then consider

$$Z_n = Y_n - \mu.$$

Then the inequality of arithmetic and geometric gives $a + b \geq 2\sqrt{ab}$

$$\begin{aligned}Z_n^4 &= Y_n^4 + 6Y_n^2\mu + \mu^4 \\ \mathrm{E}\left[Z_n^4\right] &\leq C + 6\sigma^2\mu + \mu^4 < \infty.\end{aligned}$$

Thus the proof based on Borel-Cantelli lemma is still valided.

5. (*Poisson process*) A simple *Poisson arrival process* is a sequence of times $0 = T_0 < T_1 < T_2 < \cdots$. The *inter-arrival* times $S_k = T_k - T_{k-1}$ are independent exponential random variables. The *intensity* parameter, λ, is the parameter in the exponential distribution $S_k \sim \lambda e^{-\lambda s}$, $S_k > 0$. The *counting function*[1]is $N_t = k$ if $T_k < t$ and $T_{k+1} \geq t$. Either the counting process or the arrival times are called the Poisson process. The counting process jumps from k to $k+1$ at time T_k. Therefore, it is sometimes called a *jump process*.

    (a) Derive the probability density, $f_k(t)$, of T_k. Hint:

    $$\mathrm{P}(T_k \in (t, t+dt)) = \int_{t'=0}^{t} f_{k-1}(t')\, \mathrm{P}(T_k \in (t, t+dt) \mid T_{k-1} = t')\; dt' \,.$$

[1] The inequality/equality choice makes the process N_t a *cadlag* process, more properly *càdlàg*, a French abbreviation of "continue à droite, limite à gauche", which translates to "continuous on the right, limit on the left". If you don't know French, you can remember droite, which is related to the English word "right" (both as in "rights" and as a direction), and gauche is an English word related to being clumsy (or inappropriate), which is how it is for many of us with our left hand.

Write this in terms of the S_k density, figure out the integrals, starting with $f_1(t) = \lambda e^{-\lambda t}$, then moving to $f_2(t)$, f_3, etc., until you see the pattern.

Sol: Note that for $t > 0$ and dt small enough, $\mathbb{P}(T_k \in (t, t + dt))$ means exactly $k-1$ arrives in $[0, t)$, and exactly one point in $[t, t+dt)$. Let $k = 1$, then

$$\begin{aligned}
\mathbb{P}(T_2 \in (t, t + dt)) &= \int_0^t f_1(t')\, \mathbb{P}(T_2 \in (t, t + dt) \mid T_1 = t')\ dt' \\
&= \int_0^t \lambda e^{-\lambda t'}\, \mathbb{P}(S_2 \in (t - t', t - t' + dt))\ dt' \\
&= \int_0^t \lambda e^{-\lambda t'}\ \lambda e^{-\lambda(t-t')}\, dt' \\
&= \lambda^2 t e^{-\lambda t}.
\end{aligned}$$

So $f_2(t) = \lambda^2 t e^{-\lambda t}$. Similarly,

$$\begin{aligned}
f_3(t) &= \int_0^t f_2(t')\, \mathbb{P}(T_3 \in (t, t + dt) \mid T_2 = t')\ dt' \\
&= \int_0^t \lambda^2 t' e^{-\lambda t'}\, \mathbb{P}(S_3 \in (t - t', t - t' + dt))\ dt' \\
&= \lambda^3 e^{-\lambda t} \int_0^t t' dt' \\
&= \lambda^3 e^{-\lambda t} \frac{t^{(3-1)}}{(3-1)!}.
\end{aligned}$$

We may couclude that

$$f_k(t) = \lambda^k \frac{t^{(k-1)}}{(k-1)!} e^{-\lambda t}.$$

(b) Derive a formula for $p_n(t) = \mathrm{P}(N_t = n)$. This is the *Poisson* distribution. Check that your formula satisfies $\sum_0^\infty p_n(t) = 1$. This involves the Taylor series formula for the exponential. Hint: $p_0(t) = \mathrm{P}(T_1 > t)$, $p_1(t) = \mathrm{P}(T_1 < t < T_2)$, etc. Look for the pattern. Prove it by induction.

Sol: By induction,

$$\begin{aligned} p_0(t) &= \mathbb{P}(T_1 > t) = \int_t^\infty \lambda e^{-\lambda s} ds = e^{-\lambda t} \\ p_1(t) &= \mathbb{P}(T_1 < t < T_2) \\ &= \mathbb{P}(T_2 > t) - \mathbb{P}(T_1 > t) \\ &= \left(\lambda^2 \int_t^\infty s e^{-\lambda s} ds\right) - e^{-\lambda t} \\ &= -e^{-\lambda t}(-1 - \lambda t) - e^{-\lambda t} \\ &= (\lambda t) e^{-\lambda t} \end{aligned}$$

So we may see the pattern, $p_n(t) = \mathbb{P}(T_{n+1} > t) - \mathbb{P}(T_n > t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

(c) Introduce a small time increment Δt and a probability $p_{\Delta t} = \lambda \Delta t$ ($p_{\Delta t} < 1$ for Δt small enough). Define $t_j = j\Delta t$ and independent random variables $Y_j = 1$ with probability $p_{\Delta t}$ and $Y_j = 0$ otherwise. Define

$$N_t^{\Delta t} = \sum_{t_j < t} Y_j \ .$$

Show that for each t,

$$N_t^{\Delta t} \xrightarrow{\mathcal{D}} N_t \quad \text{as} \quad \Delta t \to 0 \ .$$

Hint: The distribution of $N_t^{\Delta t}$ is binomial. The limit $\Delta t \to 0$ is easy for $p_n^{\Delta t}(t)$.

Sol: Recall that the definition of exponential is given by

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n .$$

Also floor$(x) = \lfloor x \rfloor =$ the largest integer not greater than x. Since we have $t_j = j\triangle t < t$, we are summing up to $\left\lfloor \frac{t}{\triangle t} \right\rfloor$. Notice that from the definition of $N_t^{\triangle t}$, we know it is the sum of $\left\lfloor \frac{t}{\triangle t} \right\rfloor$ independent Bernoulli trials with probability $p\triangle t$. We may write it as $N_t^{\triangle t} \sim B\left(\left\lfloor \frac{t}{\triangle t} \right\rfloor, p_{\triangle t} = \lambda \triangle t\right)$, in which the density is given by

$$\begin{pmatrix} \left\lfloor \frac{t}{\triangle t} \right\rfloor \\ k \end{pmatrix} (\lambda \triangle t)^k (1 - \lambda \triangle t)^{n-k}.$$

Besides, one can easily find the characteristic function is

$$\begin{aligned} \left(1 - \lambda \triangle t + \lambda \triangle t e^{it}\right)^{\left\lfloor \frac{t}{\triangle t} \right\rfloor} &= \left(1 + \lambda \triangle t \left(e^{it} - 1\right)\right)^{\left\lfloor \frac{t}{\triangle t} \right\rfloor} \\ &\to \exp\left(\lambda \left(e^{it} - 1\right)\right) \end{aligned}$$

as $\triangle t \to 0$. One recognize it's the characteristic function of $p_n^{\triangle t}(t)$. According to Lévy's continuity theorem: the sequence of random variables $\{X_n\}$ converges in distribution to X if and only if the sequence of corresponding characteristic functions converges pointwise to the characteristic function of X.

(d) Assuming that $N_{[0,T]}^{\Delta t} \xrightarrow{\mathcal{D}} N_{[0,T]}$ as $\Delta t \to 0$, show that the Poisson process has the independent increments property. Hint: this is a statement about discrete probabilities that you can check by using those probabilities, and figuring out what happens if t is not one of the t_j.

Sol: Let us consider the case t is one of the t_j first, then the characteristic functions of $N_{t_i} - N_{t_j}$ is

$$\exp\left(\lambda\left(e^{i(t_i - t_j)} - 1\right)\right)$$

(e) Show that N_t is a Markov process.

Sol: Recall that let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F})_{t\geq 0})$ be a stochastic basis and $X = (X_t)_{t\geq 0}$, $X_t : \Omega \to \mathbb{R}$, be a stochastic process. The process X is called a *Markov process* provided that X is adapted and for all $s, t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$ one has that

$$\mathbb{P}\left(X_{s+t} \in B | \mathcal{F}_s\right) = \mathbb{P}\left(X_{s+t} \in B | \sigma\left(X_s\right)\right) \quad a.s.$$

i. Standard Brownian motion is Markov and a Martingale,
ii. Brownian motion with drift is Markov but not a Martingale
iii. The moving average of a Brownian Motion is a Martingale but not Markov
iv. A Poisson Process is Markov but not a Martingale

(f) The *compensated* Poisson arrival process is $M_t = N_t - \lambda t$. Show that this is a martingale, which means that if $s > t$, then

$$\mathrm{E}[\, M_s \mid \mathcal{F}_t] = M_t \ .$$

Sol: Notice thta a Poisson process is not a martingale since $\mathbb{E}[N_t] = \lambda t$, which is increasing. Consider a compensated Poisson arrival process M_t, then

$$\begin{aligned}
\mathbb{E}\left[M_s - M_t | \mathcal{F}_t\right] &= \mathbb{E}\left[N_s - N_t | \mathcal{F}_t\right] - \lambda (s - t) \\
&= \mathbb{E}\left[N_s - N_t\right] - \lambda (s - t) \\
&= \mathbb{E}\left[N_{s-t}\right] - \lambda (s - t) \\
&= 0.
\end{aligned}$$

(g) The standard Poisson process has intensity, or *arrival rate*, $\lambda = 1$. Show that

$$\mathrm{E}\big[\, \Delta M^2 \mid \mathcal{F}_t \big] = \Delta t + \text{ (smaller)} \quad \text{as } \Delta t \to 0\ .$$

As usual, $\Delta M = M_{t+\Delta t} - M_t$. Compare these to comparable facts about standard Brownian motion (martingale, $\mathrm{E}\big[\, \Delta W^2 \mid \mathcal{F}_t \big]$). Conclude that Brownian motion is not the unique process with these properties.

Sol: Let $\Delta M = M_{t+\Delta t} - M_t$, then

$$\begin{aligned}
\mathbb{E}\left[\triangle M^2 \middle| \mathcal{F}_t \right] &= \mathbb{E}\left[M_{t+\Delta t}^2 - 2M_t M_{t+\Delta t} + M_t^2 \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[\left(N_{t+\Delta t} - \lambda(t+\Delta t)\right)^2 - 2\left(N_t - \lambda t\right)\left(N_{t+\Delta t} - \lambda(t+\Delta t)\right) + \left(N_t - \lambda t\right)^2 \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[\left(N_{t+\Delta t} - N_t\right)^2 \middle| \mathcal{F}_t \right] + 2\lambda t \mathbb{E}\left[N_{t+\Delta t} - N_t \middle| \mathcal{F}_t \right] \\
&- 2\lambda(t+\Delta t)\mathbb{E}\left[\left(N_{t+\Delta t} - N_t\right) \middle| \mathcal{F}_t \right] + \lambda^2 \Delta t^2 \\
&= \lambda \Delta t + 2\lambda^2 t \Delta t - 2\lambda^2 (t+\Delta t) \triangle t + \lambda^2 \Delta t^2 \\
&= \lambda \Delta t - \lambda^2 \Delta t^2 \\
&= \lambda \Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

Since we set $\lambda = 1$, this proved the claim.

(h) Calculate the scaling of $\mathrm{E}\big[\, \Delta W^4 \mid \mathcal{F}_t \big]$ and $\mathrm{E}\big[\, \Delta M^4 \mid \mathcal{F}_t \big]$ with Δt as $\Delta t \to 0$.

Sol: First we notice that $\mathbb{E}\left[\Delta W^4 \middle| \mathcal{F}_t \right]$