Assignment 6, due November 12

Corrections: (none yet.)

1. Suppose $X_t = (X_{1,t}, X_{2,t}, \ldots, X_{n,t})$, where the $X_{k,t}$ are independent standard Brownian motions. The distance of X_t from the origin is

$$R_t = \left(X_{1,t}^2 + \cdots + X_{n,t}^2\right)^{1/2} \ .$$

We calculate dynamics of the *Bessel process* R_t. This may be found in Wikipedia, but please try to do the problem independently.

(a) Suppose $f(x_1, \ldots, x_n)$ is some smooth function, and the $X_{k,t}$ are independent standard Brownian motions. Derive Ito's lemma

$$df_t = \sum_{k=1}^{n} \partial_{x_j} f \, dX_j + \frac{1}{2} \sum_{j=1}^{n} \partial_{x_j}^2 f \, dt \ .$$

This is written in vector notation as

$$df_t = \nabla f \cdot dX_t + \tfrac{1}{2} \triangle f \, dt \ .$$

The *Laplace operator*, or *Laplacian*, acting on f is

$$\triangle f = \sum_{j=1}^{n} \partial_{x_j}^2 f \ .$$

In particular, $\mathrm{E}[\, df_t \mid \mathcal{F}_t] = \frac{1}{2} \triangle f(X_t) \, dt$.

Sol: Define $\Delta f = f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) - f(x_1, \ldots, x_n)$. The Taylor series, up to the order we need, we calculate the Taylor expansion for smooth enough f,

$$\Delta f = \sum_{j=1}^{n} \partial_{x_j} f \Delta x_j + \frac{1}{2} \sum_{j=1}^{n} \partial_{x_j}^2 f \left(\Delta x_j\right)^2 + \text{(small)},$$

or in differential form

$$df_t = \nabla f \cdot dX_t + \frac{1}{2} \triangle f dt.$$

Notice that we replace Δx_j^2 by dt since the assumption of i.i.d standard BMs. In particular, since the dX_t term is of expected value vanished, we must have

$$\mathbb{E}\left[\, df_t \middle| \, \mathcal{F}_t\right] = \frac{1}{2} \triangle f(X_t) dt,$$

as claimed.

(b) For $f(x) = |x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$, calculate ∇f and $\triangle f$. Hint: ∇f points directly away from the origin (why?). For $\partial_{x_j}^2 f$, you have to use the chain rule, twice.

Sol: For $j = 1, \ldots, n$, we compute the first and second derivatives,

$$\begin{aligned}
\frac{\partial |x|}{\partial x_j} &= \frac{\partial \left(x_1^2 + \cdots + x_n^2\right)^{1/2}}{\partial x_j} \\
&= \frac{2x_j}{2\left(x_1^2 + \cdots + x_n^2\right)^{1/2}} \\
&= \frac{x_j}{|x|} \\
\frac{\partial^2 |x|}{\partial x_j^2} &= \frac{|x|^2 - x_j^2}{|x|^3}
\end{aligned}$$

In vector form, $\nabla f = \frac{x}{|x|}$, and

$$\begin{aligned}
\triangle f &= \sum_{j=1}^{n} \partial_{x_j}^2 f \\
&= \frac{n}{|x|} - \frac{1}{|x|^3} \sum_{j=1}^{n} x_j^2 \\
&= \frac{n}{|x|} - \frac{1}{|x|^3} |x|^2 \\
&= \frac{n-1}{|x|}.
\end{aligned}$$

(c) For $f(x) = |x|$, calculate $\mathrm{E}[\, dR_t \mid \mathcal{F}_t] = a(R_t)dt$ and $\mathrm{E}\left[\, dR_t^2 \mid \mathcal{F}_t\right] = \mu(R_t)dt$.

Sol: Considering $R_t = \left(X_{1,t}^2 + \cdots + X_{n,t}^2\right)^{1/2}$, then

$$\begin{aligned}
dR_t &= \nabla R_t \cdot dX_t + \frac{1}{2}\triangle R_t dt. \\
&= \frac{X_t}{R_t} \cdot dX_t + \frac{n-1}{2R_t} dt.
\end{aligned}$$

Notice that the term

$$\begin{aligned}
\frac{X_t}{R_t} \cdot dX_t &= \frac{(X_{1,t} + \cdots + X_{n,t})}{\left(X_{1,t}^2 + \cdots + X_{n,t}^2\right)^{1/2}} dW_t \\
&= dW_t,
\end{aligned}$$

and thus we can rewrite dR_t to be

$$dR_t = \frac{n-1}{2R_t} dt + dW_t.$$

Now it's obvious that,

$$\mathbb{E}\left[dR_t\middle|\,\mathcal{F}_t\right] = \frac{n-1}{2R_t}dt = a\left(R_t\right)dt$$
$$\mathbb{E}\left[dR_t^2\middle|\,\mathcal{F}_t\right] = dt = \mu(R_t)dt,$$

namely $a\left(R_t\right) = \frac{n-1}{2R_t}$ and $\mu\left(R_t\right) = 1$.

2. (*Forward equation*) Suppose X_t is a diffusion process and $u(x,t)$ is the probability density of X_t (as a function of x). Then u satisfies a *forward equation*, which is a PDE. Early in the course we saw that if X_t is Brownian motion, then u satisfies the heat equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u \ .$$

This exercise suggests how to find the forward equation for more general diffusions. The full derivation by this method is time consuming, so we will do a special case, the case of *additive noise.* Additive noise means that the coefficient $\mu(x)$ in the infinitesimal variance (equation (2) of Week 7) is independent of x.

The first step is to approximate the process X_t by a discrete time process $X_j^{\Delta t}$. We want $X_j^{\Delta t} \approx X_{t_j}$, where $t_j = j\Delta t$ as always. The approximation will take the form

$$X_{j+1}^{\Delta t} = X_j^{\Delta t} + a(X_j^{\Delta t})\Delta t + bZ_j \ , \tag{1}$$

where the Z_j are i.i.d. $Z_j \sim \mathcal{N}(0,1)$, and the coefficient b is yet to be determined. We want the approximation to have the properties that

$$\mathrm{E}\left[\,\Delta X_j^{\Delta t} \mid \mathcal{F}_j\right] = a(X_j^{\Delta t})\Delta t \ , \tag{2}$$

and

$$\mathrm{var}\left(\Delta X_j^{\Delta t} \mid \mathcal{F}_j\right) = \mu\Delta t \ . \tag{3}$$

Here, μ is the constant value of $\mu(x)$ above. The formula (1) satisfies the drift condition (2) automatically. Computing from (1) gives $\mathrm{var}\left(\Delta X_j^{\Delta t} \mid \mathcal{F}_j\right) = b^2$. This gives (3) if $b = \sqrt{\mu\Delta t}$. Therefore, our approximation is

$$X_{j+1}^{\Delta t} = X_j^{\Delta t} + a(X_j^{\Delta t})\Delta t + \sqrt{\mu\Delta t}Z_j \ . \tag{4}$$

This approximation satisfies the continuity condition

$$\mathrm{E}\left[\,\left(\Delta X_j^{\Delta t}\right)^4 \mid \mathcal{F}_j\right] \leq C\Delta t^2 \ .$$

It is possible to prove that approximations with these properties converge to the exact stochastic process X_t in the sense of distributions.

Let $u_j(x)$ be the probability density of $X_j^{\Delta t}$. This should be an approximation of the probability density of X_{t_j}, which is

$$u_j(x) \approx u(x,t_j) \ .$$

The forward equation PDE that $u(x,t)$ satisfies has the form

$$\partial_t u = L^* u \ , \tag{5}$$

where L^* is a *differential operator* in the x variable. We will find L^* by finding a formula

$$u_{j+1}(x) = u_j(x) + \Delta t L^* u_j(x) + O(\Delta t^{3/2}) \ . \tag{6}$$

The technique will be to derive an integral formula for u_{j+1} in terms of u_j, and then to derive (6) from the integral formula by approximating the integral.

(a) Let $v_j(x,y)$ be the joint probability density of $(X_j^{\Delta t}, X_{j+1}^{\Delta t})$. Here x is the $X_j^{\Delta t}$ variable and y is the $X_{j+1}^{\Delta t}$ variable. For example,

$$\mathrm{P}\big(X_j^{\Delta t} > X_{j+1}^{\Delta t}\big) = \int_{-\infty}^{\infty} \int_{y=-\infty}^{x} v_j(x,y)\, dy dx \ .$$

Write a formula for $v_j(x,y)$ in terms of $u_j(x)$. This is done using (4) and the formula for a Gaussian density. Use this to find a formula of the form

$$u_{j+1}(y) = \int_{-\infty}^{\infty} K(x,y,\Delta t) u_j(x)\, dx \ , \tag{7}$$

I with a simple Gaussian explicit formula for K.

Sol: First notice that the linear transform $a(x)\Delta t + \sqrt{\mu \Delta t}\mathcal{N}(0,1)$ is distributed $\mathcal{N}(a(x)\Delta t, \mu\Delta t) := Z$. By independence, the joint probability density of $(X_j^{\Delta t}, X_{j+1}^{\Delta t})$ is given by

$$v_j(x,y) = u_j(x) \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(- \frac{(y - x - a(x)\Delta t)^2}{2\mu\Delta t} \right) .$$

Another point of view is that

$$\begin{aligned}
\mathbb{P}\left(X_j^{\Delta t} \le x, X_{j+1}^{\Delta t} \le y\right) &= \mathbb{P}\left(X_j^{\Delta t} \le x, X_j^{\Delta t} + a(X_j^{\Delta t})\Delta t + \sqrt{\mu\Delta t} Z_j \le y\right) \\
&= \mathbb{P}\left(X_j^{\Delta t} \le x, Z \le \frac{y - X_j^{\Delta t} - a(X_j^{\Delta t})\Delta t}{\sqrt{\mu \Delta t}}\right) \\
&= \int_{-\infty}^{x} \int_{-\infty}^{\frac{y - X_j^{\Delta t} - a(X_j^{\Delta t})\Delta t}{\sqrt{\mu\Delta t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} u_j(x')\, dz dx',
\end{aligned}$$

and thus

$$\begin{aligned} v_j(x,y) &= \partial_y \partial_x \mathbb{P}\left(X_j^{\Delta t} \le x, X_{j+1}^{\Delta t} \le y\right) \\ &= u_j(x) \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{\left(y - x - a\left(x\right)\Delta t\right)^2}{2\mu\Delta t}\right). \end{aligned}$$

Simply applying the *convolution formula* for the sum of independent random variables $Y = X + Z$,

$$f_Y(y) = \int f_Z(y-x) f_X(x)\, dx,$$

the density of $X_{j+1}^{\Delta t}$ is given by

$$u_{j+1}(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{\left(y - x - a\left(x\right)\Delta t\right)^2}{2\mu\Delta t}\right) u_j(x)\, dx,$$

and thus

$$K(x,y,\Delta t) = \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{\left(y - x - a\left(x\right)\Delta t\right)^2}{2\mu\Delta t}\right).$$

(b) A "warm up problem" is a problem you know the answer to already that you can use to figure out this new mathematical machinery. In the present case, you know that if X_t is Brownian motion, the PDE (5) should be the heat equation. This part of the exercise shows that (6) is satisfied with $L^* u = \frac{1}{2}\partial_x^2 u$. In the integral (7), make a Taylor expansion of $u_j(x)$ for x near y. This may be done in two steps, first to a change of variables $x \leftarrow z = x - y$, then make the Taylor expansion

$$\begin{aligned} u_j(y+z) &= u_j(y) + O\left(|z|\right) \quad \text{or} \\ u_j(y+z) &= u_j(y) + \partial_x u_j(y) z + O\left(z^2\right) \quad \text{or} \\ u_j(y+z) &= u_j(y) + \partial_x u_j(y) z + \tfrac{1}{2}\partial_x^2 u_j(y) z^2 + O\left(|z|^3\right) \quad \text{or} \\ u_j(y+z) &= u_j(y) + \partial_x u_j(y) z + \tfrac{1}{2}\partial_x^2 u_j(y) z^2 + \tfrac{1}{6}\partial_x^3 u_j(y) z^3 + O\left(z^4\right) \end{aligned}$$

You know you have gone far enough when the remainder term contributes something smaller than $O(\Delta t)$. That is, $\int K(y+z, y, \Delta t)\, |z|^p\, dz = O(\Delta t^{3/2})$.

Sol: In this warm up problem, we consider the case X_t is Brownian motion, namely $a(X_t) = 0$, $\mu = 1$, and $K(x,y,\Delta t) = \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{(y-x)^2}{2\mu\Delta t}\right)$. According to the hint, first do a change of variables $x \leftarrow z = x - y$, then make the Taylor expansion for x around y,

$$
\begin{aligned}
u_{j+1}(y) &= \int_{-\infty}^{\infty} K(y+z, y, \Delta t)\left[u_j(y) + \partial_x u_j(y) z + \tfrac{1}{2}\partial_x^2 u_j(y) z^2 + \tfrac{1}{6}\partial_x^3 u_j(y) z^3 + O\left(z^4\right)\right] dx \\
&= u_j(y)\int_{-\infty}^{\infty} K(x, y, \Delta t) dx + \partial_x u_j(y) \int_{-\infty}^{\infty} K(y+z, y, \Delta t) z\, dz \\
&\quad + \tfrac{1}{2}\partial_x^2 u_j(y) \int_{-\infty}^{\infty} K(y+z, y, \Delta t) z^2\, dz + ... \\
&= u_j(y)\mathbb{E}\left[Z^0\right] + \partial_x u_j(y)\mathbb{E}\left[Z^1\right] + \tfrac{1}{2}\partial_x^2 u_j(y)\mathbb{E}\left[Z^2\right] + \tfrac{1}{6}\partial_x^3 u_j(y)\mathbb{E}\left[Z^3\right] + O\left(\mathbb{E}\left[Z^4\right]\right) \\
&= u_j(y)\cdot 1 + \tfrac{1}{2}\partial_x^2 u_j(y)\cdot \Delta t + O\left(3\Delta t^{3/2}\right) \\
&= u_j(y) + \Delta t L^* u_j(y) + O(\Delta t^{3/2}).
\end{aligned}
$$

Recall that for mean zero Gaussian,

$$
\mathbb{E}\left[Z^p\right] = \begin{cases} 0 & \text{if p is odd} \\ \sigma^p (p-1)!! & \text{if p is even} \end{cases}
$$

where $n!!$ is double factorial defined by

$$
(2n)!! = 2^n n!.
$$

To conclude, $L^* u = \frac{1}{2}\partial_x^2 u$ as claimed.

(c) Do the problem now under the hypothesis that $a(x)$ is independent of x. Let $x^*(y)$ be the x value that maximizes the integrand K over x, then let $x = x^*(y) + z$. This should give $u_{j+1}(y) = u_j(x^*(y)) + \frac{1}{2}\partial_x^2 u_j(x^*))$. But $x^*(y) = y + O(\Delta t)$, so you can write express $u_j(x^*(y))$ as a Taylor expansion about $u_j(y)$ and ignore terms beyond $O(\Delta t)$.

Sol: First of all, it's obvious that K has maximum when

$$
y - x^*(y) - a\left(x^*(y)\right)\Delta t = 0,
$$

and we set $x^*(y)$ satisfy this equation. Now consider $x = x^*(y) + z$, then

$$
\begin{aligned}
K(x^*(y) + z, y, \Delta t) &= \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{\left(y - x^*(y) - z - a\left(x^*(y)\right)\Delta t\right)^2}{2\mu\Delta t}\right) \\
&= \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left(-\frac{z^2}{2\mu\Delta t}\right).
\end{aligned}
$$

Therefore, considering the Taylor expansion,

$$
u_j(x^*(y) + z) = u_j(x^*(y)) + \partial_x u_j(x^*(y)) z + \tfrac{1}{2}\partial_x^2 u_j(x^*(y)) z^2,
$$

we have

$$
\begin{aligned}
u_{j+1}(y) &= \int_{-\infty}^{\infty} K(x,y,\Delta t) u_j(x)\,dx \\
&= \int_{-\infty}^{\infty} K(x^*(y)+z,y,\Delta t) u_j(x^*(y)+z,)\,dz \\
&= \frac{1}{\sqrt{2\pi\mu\Delta t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\mu\Delta t}} u_j(x^*(y)+z)\,dz \\
&= \frac{1}{\sqrt{2\pi\mu\Delta t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\mu\Delta t}} \left[u_j(x^*(y)) + \partial_x u_j(x^*(y))z + \tfrac{1}{2}\partial_x^2 u_j(x^*(y))z^2\right] dz \\
&= u_j(x^*(y)) + \tfrac{1}{2}\partial_x^2 u_j(x^*(y))\mu\Delta t + \mathcal{O}(\Delta t^{3/2}) \\
&= \left[u_j(y) - a\Delta t\partial_x u_j(y) + \mathcal{O}(\Delta t^2)\right] + \tfrac{1}{2}\left[\partial_x^2 u_j(y) + O(\Delta t)\right]\mu\Delta t + \mathcal{O}(\Delta t^{3/2}) \\
&= u_j(y) + \left(-a\partial_x u_j(y) + \tfrac{1}{2}\partial_x^2 u_j(y)\mu\right)\Delta t + \mathcal{O}(\Delta t^{3/2}),
\end{aligned}
$$

which gives

$$L^* u = \left(-a\partial_x + \tfrac{1}{2}\partial_x^2 \mu\right) u.$$

(d) The last step is the hardest. In K you find the expression somewhere $(y - x - a(x)\Delta t)^2$. We want to integrate over x. We do not need to do the integral exactly, so we can make approximations in this expression if they do not change the answer up to $O(\Delta t)$. Make a Taylor series approximation to a about y (y is a parameter in the integration) $a(x) = a(y) + a'(y)(x-y) + \cdots$ (you figure out how far you need to go). You should get something involving

$$\frac{u_j(x^*)}{1 \pm a'(y)\Delta t} + \cdots$$

This is

$$u_j(x^*)\left(1 \mp a'(y)\Delta t\right),$$

which gives all the contributions of order Δt. The answer is supposed to be $L^* u_j(x) = -\partial_x\left(a(x)u_j(x)\right) + \frac{1}{2}\partial_x^2 u_j(x)$.

Sol: Make a Taylor series approximation to a about y, that is

$$y - x - \Delta t\left[a(y) + a'(y)(x-y) + \mathcal{O}\left(|x-y|^2\right)\right]$$

Since $|y-x|^2 = \left|X_{j+1}^{\Delta t} - X_j^{\Delta t}\right|^2 = \mathcal{O}\left(\Delta t^2\right)$, ignoring up to $\mathcal{O}\left(\Delta t^2\right)$ and then solving for the maximum of K by considering

$$y - x^* - \Delta t\left[a(y) + a'(y)(x^*-y)\right] = 0$$

yields

$$\begin{aligned} x^* &= \frac{y - \Delta t \left[a(y) - a'(y)y\right)]}{1 + a'(y)\Delta t} \\ &= \frac{(1 + a'(y)\Delta t)y - \Delta t a(y)}{1 + a'(y)\Delta t} \\ &= y - \frac{\Delta t a(y)}{1 + a'(y)\Delta t} \\ &= y - \Delta t a(y) \left(1 - a'(y)\Delta t\right) \end{aligned}$$

To make our life simpler, denote $(1 + a'(y)\Delta t) = A(y, \Delta t)$. Then let $z = \frac{x - x^*}{A}$, note that $x = x^* + Az$ and thus

$$\begin{aligned} K(x, y, \Delta t) &= \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left[-\frac{\left(y - x^* - zA - \Delta t \left[a(y) + a'(y)(x^* - y)\right]\right)^2}{2\mu\Delta t}\right] \\ &= \frac{1}{\sqrt{2\pi\mu\Delta t}} \exp\left[-\frac{(Az)^2}{2\mu\Delta t}\right]. \end{aligned}$$

So we have

$$\begin{aligned} u_{j+1}(y) &= \int_{-\infty}^{\infty} K(x^* + Az, y, \Delta t) u_j(x^* + Az)\, A dz \\ &= \int_{-\infty}^{\infty} \frac{A^2}{\sqrt{2\pi\mu\Delta t}} \exp\left[-\frac{Az^2}{2\mu\Delta t}\right] \frac{u_j(x^* + Az)}{A}\, dz. \end{aligned}$$

Also notice that we can approximate the last term of the above integrand

$$\frac{u_j(x^* + Az)}{A} = \left[u_j(x^*) + \partial_x u_j(x^*)Az + \tfrac{1}{2}\partial_x^2 u_j(x^*)A^2 z^2\right] \left[(1 - a'(y)\Delta t) + \mathcal{O}(\Delta t^2)\right],$$

which gives

$$\begin{aligned} u_{j+1}(y) &= \left[u_j(x^*) + \partial_x u_j(x^*)Az + \tfrac{1}{2}\partial_x^2 u_j(x^*)A^2 z^2\right] \\ &\quad u_j(x^*) \left(1 - a'(y)\Delta t\right) + \tfrac{1}{2}\partial_x^2 u_j(y) \frac{(1 - a'(y)\Delta t)}{\left(1 + a'(y)\Delta t\right)^2} \mu\Delta t + \mathcal{O}(\Delta t^{3/2}) \\ &= u_j(x^*) \left(1 - a'(y)\Delta t\right) + \left(\tfrac{1}{2}\partial_x^2 u_j(y) + O(\Delta t)\right) \mu\Delta t + \mathcal{O}(\Delta t^{3/2}) \\ &= \left[u_j(y) - a(y)\Delta t \partial_x u_j(y)\right] \left(1 - a'(y)\Delta t\right) + \tfrac{1}{2}\left[\partial_x^2 u_j(y) + O(\Delta t)\right] \mu\Delta t + \mathcal{O}(\Delta t^{3/2}) \\ &= u_j(y) - a(y)\Delta t \partial_x u_j(y) + u(y)a'(y)\Delta t + \tfrac{1}{2}\mu\partial_x^2 u_j(y)\Delta t + \mathcal{O}(\Delta t^{3/2}) \\ &= u_j(y) + \Delta t \left(-\partial_y \left(a(y)u_j(y)\right) + \frac{1}{2}\partial_y^2 u_j(y)\right) + O(\Delta t^{3/2}). \end{aligned}$$

So we found that

$$L^* u_j(x) = -\partial_x \left(a(x)u_j(x)\right) + \frac{1}{2}\partial_x^2 u_j(x),$$

as claimed before.