Stochastic Calculus, Courant Institute, Fall 2012
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2012/index.html
Always check the class message board on the blackboard site from home.nyu.edu before doing any work on the assignment.

Assignment 7, due December 10

Corrections: (none yet.)

1. Suppose we want to evaluate $A = \mathrm{E}\left[e^{-X_T^2/2}\right]$ where X_t is a standard Brownian motion starting from $X_0 = 0$. One approach is to simulate N Brownian motion paths and use the estimator

$$\widehat{A} = \frac{1}{N}\sum_{k=1}^{N} e^{-X_{k,T}^2/2} \ . \tag{1}$$

Another approach is to simulate the Ornstein Uhlenbeck process

$$dX_t = -\gamma X_t dt + dW_t \ .$$

Then there is a change of measure formula $L(X)$ so that

$$A = \mathrm{E}_{\mathrm{OU}}\left[e^{-X_{k,T}^2/2} L(X_{[0,T]})\right] \ . \tag{2}$$

Another way to estimate A is to simulate N Ornstein Uhlenbeck paths use

$$\widehat{A} = \frac{1}{N}\sum_{k=1}^{N} e^{-X_{k,T}^2/2} L(X_{k,[0,T]}) \ . \tag{3}$$

The$\mathrm{P}(\tau \geq t)$ $X_0/\mathrm{P}(\tau > t)$second approach is more complicated, but it could be a better estimator for large T.

(a) Write an analytic formula for A as a function of T.
Sol: Straightforward computation,

$$\begin{aligned} A &= \mathbb{E}\left[e^{-X_T^2/2}\right] \\ &= \frac{1}{\sqrt{2\pi T}}\int_{\mathbb{R}} \exp\left[-\frac{x^2}{2} - \frac{x^2}{2T}\right] dx \\ &= \frac{1}{\sqrt{T+1}}\frac{\sqrt{T+1}}{\sqrt{2\pi T}}\int_{\mathbb{R}} \exp\left[-\frac{(T+1)x^2}{2T}\right] dx \\ &= \frac{1}{\sqrt{T+1}}. \end{aligned}$$

(b) Write a formula for the variance of the estimator (1).
Sol: The variance the estimator is given by

$$\begin{aligned}
\mathrm{Var}\left[\hat{A}\right] &= \mathbb{E}\left[\hat{A}^2\right] - \left(\mathbb{E}\left[\hat{A}\right]\right)^2 \\
&= \mathbb{E}\left[\left(\frac{1}{N}\sum_{k=1}^{N} e^{-X_{k,T}^2/2}\right)^2\right] - \left(\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\sqrt{T+1}}\right)^2 \\
&= \mathbb{E}\left[\frac{1}{N^2}\sum_{k=1}^{N} e^{-X_{k,T}^2} + \frac{2}{N^2}\sum_{k>j} e^{-(X_{k,T}^2+X_{j,T}^2)/2}\right] - \frac{1}{T+1} \\
&= \frac{1}{N^2}\sum_{k=1}^{N}\mathbb{E}\left[e^{-X_{k,T}^2}\right] + \frac{2}{N^2}\sum_{k>j}\mathbb{E}\left[e^{-X_{k,T}^2/2}\right]\mathbb{E}\left[e^{-X_{j,T}^2/2}\right] - \frac{1}{T+1} \\
&= \frac{1}{N}\frac{1}{\sqrt{2T+1}} + \frac{\cancel{2}}{N^2}\frac{N^2-N}{\cancel{2}}\cancel{\frac{1}{T+1}} - \cancel{\frac{1}{T+1}} \\
&= \frac{1}{N\sqrt{2T+1}} - \frac{1}{N(T+1)} \\
&= \frac{\sqrt{2T+1}(T+1) - (2T+1)}{N(2T+1)(T+1)}
\end{aligned}$$

(c) Use these to show that the relative accuracy of the Monte Carlo estimator gets worse as T increases. Give an intuitive explanation for this in terms of the distribution of X_T and the range of values of X_T that contribute most to A. Make your explanation quantitative (giving the right power of T) if you can.

(d) Write a formula for L in (2) that gives the correct A. This is an application of Girsanov's formula.
Sol: Applying Girsanov's formula

$$\begin{aligned}
L\left(X_{[0,T]}\right) &= \exp\left[-\gamma\int_0^T X_t dX_t - \frac{1}{2}\int_0^T \gamma^2 X_t^2 dt\right] \\
&= \exp\left[-\gamma\int_0^T X_t dX_t\right]\exp\left[-\frac{\gamma^2}{2}\int_0^T X_t^2 dt\right]
\end{aligned}$$

(e) That formula involves

$$Y_t = \int_0^t X_t dX_t$$

when X_t is the Ornstein Uhlenbeck process. Find an explicit expression for Y_t.
Sol: To find Y_t, we shall compute the increment square first, notice

that $dX_t^2 = dt$.

$$
\begin{aligned}
Y_t &= \int_0^t X_t dX_t \\
&= \lim_{\Delta t \to 0} \sum_{k=1}^{n-1} X_{t_k} \left(X_{t_{k+1}} - X_{t_k} \right) \\
&= \lim_{\Delta t \to 0} \frac{1}{2} \sum_{k=1}^{n-1} \left(X_{t_{k+1}}^2 - X_{t_k}^2 \right) - \frac{1}{2} \sum_{k=1}^{n-1} \left(X_{t_{k+1}} - X_{t_k} \right)^2 \\
&= \frac{1}{2} X_T^2 - \frac{1}{2} T
\end{aligned}
$$

(f) Use your answer to part (e) to find an explicit formula for A in terms of the OU process. This should agree with your answer to part (a).

Sol: First of all, we compute Now let us compute $Y_t = \int_0^t \frac{X_s^2}{2} ds$, then

$$
\begin{aligned}
dY_t &= \frac{X_t^2}{2} dt \\
&= 2X_t \left(-\gamma X_t dt + dW_t \right) \\
&= -2\gamma X_t^2 dt + 2X_t dW_t.
\end{aligned}
$$

Therefore,

$$
X_T^2 = -2\gamma \int_0^T X_t dt + 2 \int_0^T X_t dW_t.
$$

Similarly,

$$
\begin{aligned}
\int_0^T X_t dW_t &= \lim_{\Delta t \to 0} \sum_{k=1}^{n-1} X_{t_k} \left(W_{t_{k+1}} - W_{t_k} \right) \\
&= \lim_{\Delta t \to 0} \sum_{k=1}^{n-1} -X_{t_k} \left(W_{t_{k+1}} - W_{t_k} \right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}_{\mathrm{OU}}\Big[e^{-X_{k,T}^2/2}L(X_{[0,T]})\Big] &= \mathrm{E}_{\mathrm{OU}}\left[e^{-X_{k,T}^2/2}\exp\left[-\gamma\int_0^T X_t dX_t\right]\exp\left[-\gamma^2\int_0^T \frac{X_t^2}{2}dt\right]\right] \\
&= \sqrt{\frac{\gamma}{\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{2}-\gamma x^2-\gamma\left(\frac{1}{2}x^2-\frac{1}{2}T\right)-?\right]dx \\
&= \sqrt{\frac{\gamma}{\pi}}e^{\frac{\gamma T}{2}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{2}-\frac{\gamma x^2}{T}-\frac{\gamma}{2}x^2-?\right]dx \\
&= \sqrt{\frac{\gamma}{\pi T}}e^{\frac{\gamma T}{2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(T+2\gamma+\gamma T)}{2T}x^2-(\frac{\gamma T}{2}-\frac{1}{2}x^2+\frac{\gamma x^2}{2})\right]dx \\
&= \sqrt{\frac{\gamma}{\pi T}}\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma T x^2+2\gamma x^2+\gamma T x^2-\gamma T^2}{2T}\right]dx \\
&= \sqrt{\frac{\gamma}{\pi T}}\int_{-\infty}^{\infty}\exp\left[-\frac{2\gamma\,(T+1)\,x^2}{2T}\right]dx \\
&= \frac{1}{\sqrt{T+1}}\sqrt{\frac{\gamma(T+1)}{\pi T}}\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma\,(T+1)\,x^2}{T}\right]dx \\
&= \frac{1}{\sqrt{T+1}}
\end{aligned}
$$

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recall that the probability density function of the Ornstein–Uhlenbeck process is given by

$$
u_{\mathrm{OU}}(x,T)=\sqrt{\frac{\gamma}{\pi T}}\exp\left[-\frac{\gamma x^2}{T}\right].
$$

Since we know A can be written as

$$
\frac{1}{\sqrt{T+1}}=E_{OU}\Big[e^{-X_{k,T}^2/2}L(X_{[0,T]})\Big].
$$

Therefore

$$
\begin{aligned}
L\left(X_{[0,T]}\right) &= \frac{v_{X_T}}{u_{\mathrm{OU}}} \\
&= \frac{\sqrt{1/2\pi T}}{\sqrt{\gamma/\pi T}}\exp\left[-\frac{x^2}{2T}+\gamma\frac{x^2}{T}\right] \\
&= \frac{1}{\sqrt{2\gamma}}\exp\left[-\frac{(1-\gamma)x^2}{2T}\right].
\end{aligned}
$$

2. Suppose X_t is Brownian motion with $X_0=1$. Let τ be the stopping time that is the first time $X_t=0$. On previous assignments we have studied hitting probabilities.

(a) Write a formula for the probability density for X_t conditional on $\tau > t$.

Sol: First of all, we can calculate the conditional probability, $\mathbb{P}\left(X_t \le y \mid \tau > t\right)$. It's clear that for $y \le 0$, this probability is equal to zero since this process never hits the stopping position 0. Let $y > 0$, by definition the process X_t

$$\begin{aligned}
\mathbb{P}\left(X_t \le x \mid \tau > t\right) &= \mathbb{P}\left(W_t + X_0 \le x \mid \min_{s\le t} W_s + X_0 > 0\right) \\
&= \mathbb{P}\left(W_t \le x - X_0 \mid \min_{s\le t} W_s > -X_0\right) \\
&= \mathbb{P}\left(W_t \le x - X_0 \mid \max_{s\le t} W_s < X_0\right) \\
&= \frac{\mathbb{P}\left(W_t \le x - X_0,\, \max_{s\le t} W_s < X_0\right)}{\mathbb{P}\left(\max_{s\le t} W_s < X_0\right)}.
\end{aligned}$$

Note that it's obvious

$$\mathbb{P}\left(W_t \le x - X_0,\, \max_{s\le t} W_s < X_0\right) = \mathbb{P}\left(W_t \le x - X_0\right) - \mathbb{P}\left(W_t \le x - X_0,\, \max_{s\le t} W_s \ge X_0\right),$$

and we already knew that

$$\mathbb{P}\left(W_t \le x - X_0,\, \max_{s\le t} W_s \ge X_0\right) = \int_{-\infty}^{x-X_0}\int_{X_0}^{\infty} \frac{2(2u-v)}{\sqrt{2\pi t^3}} e^{-(2u-v)^2/2t} du dv.$$

To find the density, simply differentiate above with respect to x,

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbb{P}\left(W_t \le x - X_0,\, \max_{s\le t} W_s \ge X_0\right) &= \int_{-\infty}^{x-X_0}\int_{X_0}^{\infty} \frac{2(2u-v)}{\sqrt{2\pi t^3}} e^{-(2u-v)^2/2t} du dv \\
&= \int_{X_0}^{\infty} \frac{2(2u-x+X_0)}{\sqrt{2\pi t^3}} e^{-(2u-x+X_0)^2/2t} du \\
&= \int_{X_0}^{\infty} \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{2\pi t}} e^{-(2u-x+X_0)^2/2t}\right) du \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+X_0)^2}{2t}}.
\end{aligned}$$

The conditional density of X_t given that $\tau > t$ is

$$\begin{aligned}
p\left(x \middle| \tau > t\right) &= \frac{\partial}{\partial x}\mathbb{P}\left(X_t \le x \mid \tau > t\right) \\
&= \frac{1}{\mathbb{P}\left(\tau > t\right)}\left(\frac{\partial}{\partial x}\mathbb{P}\left(W_t \le x - X_0\right) - \frac{\partial}{\partial x}\mathbb{P}\left(W_t \le x - X_0, \tau \le t\right)\right) \\
&= \frac{1}{\mathbb{P}\left(\tau > t\right)}\left(\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-X_0)^2}{2t}\right) - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x+X_0)^2}{2t}\right)\right).
\end{aligned}$$

(b) Show by explicit calculation that

$$\mathrm{E}[\, X_t \mid \tau > t] = \frac{X_0}{\mathbb{P}\,(\tau > t)}.$$

Sol: It's clear that

$$\begin{aligned}
\mathbb{E}\,[X_t \mid \tau > t] &= \int_0^\infty x p\,(x|\,\tau > t)\, dx \\
&= \frac{1}{\mathbb{P}\,(\tau > t)} \frac{1}{\sqrt{2\pi t}} \int_0^\infty x \left(e^{-\frac{(x-X_0)^2}{2t}} - e^{-\frac{(x+X_0)^2}{2t}} \right) dx \\
&= \frac{1}{\mathbb{P}\,(\tau > t)} \frac{1}{\sqrt{2\pi t}} \left(\int_{-X_0}^\infty (y + X_0) e^{-\frac{y^2}{2t}}\, dy - \int_{X_0}^\infty (y - X_0) e^{-\frac{y^2}{2t}}\, dy \right) \\
&= \frac{1}{\mathbb{P}\,(\tau > t)} \left(\underbrace{\frac{1}{\sqrt{2\pi t}} \int_{-X_0}^{X_0} y e^{-\frac{y^2}{2t}}\, dy}_{\to\, 0} + X_0 \right) \\
&= \frac{X_0}{\mathbb{P}\,(\tau > t)}.
\end{aligned}$$

(c) Use the result of part (b) to show that the stopped process $X_{t\wedge\tau}$ satisfies $\mathrm{E}[\, X_{t\wedge\tau}] = X_0$.
*Sol:*From part (b), we split the expectation to two parts as follows,

$$\begin{aligned}
\mathbb{E}\,[X_{t\wedge\tau}] &= \mathbb{E}\,[X_{t\wedge\tau}|\,\tau \le t]\, \mathbb{P}\,(\tau \le t) + \mathbb{E}\,[X_{t\wedge\tau}|\,\tau > t]\, \mathbb{P}\,(\tau > t) \\
&= \underbrace{\mathbb{E}\,(X_\tau = 0\,|\,\tau \le t)}_{\to\, 0} \mathbb{P}\,(\tau \le t) + \frac{X_0}{\mathbb{P}\,(\tau > t)} \mathbb{P}\,(\tau > t) \\
&= X_0,
\end{aligned}$$

as claimed. We point out here first term of the second line vanishes simply because X_τ is the particle position when hitting 0, that is $X_\tau = 0$, and as the expectation.

3. Consider the stochastic differential equation

$$dX_t = -\gamma X_t dt + \sigma \sqrt{X_t} dW_t\ . \tag{4}$$

with $X_0 = 1$.

(a) Give a qualitative derivation of (4) by thinking of a large number of people waiting in a line. Let N_k be the number of people waiting in line at step k. Suppose N_k is a large number. At time k, everyone in the line tosses a coin, all independent, and leaves with probability ϵ. Find a scaling of ϵ and t with N so that time dt corresponds to $k \to k+1$ and the scaled N_k converges in distribution to the process (4). This just means finding a scaling factor $r(\epsilon)$ and $s(\epsilon)$ (both

powers of ϵ) so that $\mathrm{E}[\,dX_t]$ and $\mathrm{E}\left[\,dX_t^2\right]$ are both of order dt.
Sol: At time k,

$$N_{k+1} = N_k - \gamma N_k dt + \sigma\sqrt{N_k dt} Z_k.$$

Let us compute the expected value of the Cox-Ingersoll-Ross process, first write down the integral form,

$$X_t = X_0 - \gamma \int_0^t X_s ds + \sigma \int_0^t \sqrt{X_s} dW_s,$$

and then taking expectations and remind that the third term shall vanish since Ito integral is zero,

$$\mathbb{E}\left[X_t\right] = X_0 - \gamma \int_0^t \mathbb{E}\left[X_s\right] ds.$$

Differentiation yields,

$$\frac{d}{dt}\mathbb{E}\left[X_t\right] = -\gamma \mathbb{E}\left[X_t\right],$$

which implies that

$$\mathbb{E}\left[X_t\right] = e^{-\gamma t} X_0.$$

Notice that

$$\begin{aligned} \mathbb{E}\left[dX_t\right] &= -\gamma \mathbb{E}\left[X_t\right] dt = -\gamma e^{-\gamma t} X_0 dt \\ \mathbb{E}\left[dX_t^2\right] &= \frac{\sigma^2}{2}\mathbb{E}\left[X_t\right] dt = \frac{\sigma^2 e^{-\gamma t}}{2} X_0 dt. \end{aligned}$$

Taylor expansion gives

$$\begin{aligned} \mathbb{E}\left[dX_t\right] &= e^{-\gamma(t+dt)} X_0 - e^{-\gamma t} X_0 \\ &= X_0 \left(\left(1 - \gamma(t+dt) + \frac{\gamma^2 (t+dt)^2}{2!} + \ldots\right) - \left(1 - \gamma t + \frac{(\gamma t)^2}{2!} + \mathcal{O}\left(t^3\right)\right)\right) \\ &= X_0 \left(\left(-\gamma + t\gamma^2\right) dt + \mathcal{O}\left(dt^2\right)\right). \end{aligned}$$

(b) Write a program in `R` to simulate the process (4) up to time $t = 1$. Plot a histogram of the distribution of X_1 (take $\gamma = .5$ and $\sigma = 1$). Show that the histogram is incorrect if Δt is too large, but seems to have a limit as $\Delta t \to 0$.