Assignment 10, Partial. More problems to be posted tomorrow. due December 2

Corrections (none yet)

1. (*Rare events for one dimensional normals*) Let $X \sim \mathcal{N}(0,1)$ and consider the event $A = \{X \geq K\}$. This exercise shows that reweighting is both a theoretical tool to find approximate formulas for rare event probabilities and a computational tool that can be a vast improvement over vanilla Monte Carlo sampling.

   (a) (*Large deviation theory estimate*). Suppose $w(y)$ is a continuous probability density with $w(0) > 0$. Show that

   $$\lim_{\mu \to \infty} \mu \, \mathrm{E}\big[\mathbf{1}_{[0,\infty)}(Y) e^{-\mu Y}\big] = w(0) \; .$$

   Hint: Write the integral expression for the expectation value. Break the integral into the part over $[0, \varepsilon)$ and the part over $[\varepsilon, \infty)$. The first part converges to $w(0)$ as $\varepsilon \to 0$ and $\mu \to \infty$ in the right way. The second part converges to zero.

   (b) Show that $p(K) = \mathrm{P}(X \geq K) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{K} e^{-K^2/2}$ for large K. More precisely, show that

   $$\lim_{K \to \infty} K e^{K^2/2} \, \mathrm{P}(X \geq K) \; = \; \frac{1}{\sqrt{2\pi}} \; .$$

   Hint: Use a change of measure $v(x)$ that makes $X \sim \mathcal{N}(K,1)$, change variables in the integral to make $x = K$ correspond to $y = 0$, then use part (1a).

   (c) Let $Y = \mathbf{1}_{x \geq K}(X)$. Show that $\mathrm{var}(Y) \approx p(K)$ for large K.

   (d) Let $\widehat{M_0}$ be the vanilla estimate of $p(K)$ given by

   $$\widehat{M_0} = \frac{1}{N} \# \left\{ X_j \geq K \;\; \text{with } 1 \leq j \leq N \right\} \; .$$

   Let $\sigma_0(N)$ be the standard deviation of $\widehat{M_0}$. The *relative accuracy* of $\widehat{M_0}$ is

   $$\frac{\widehat{M_0} - p(K)}{p(K)} \sim \frac{\sigma_0(N)}{p(K)} \; .$$

   Show that when K is large and N is large enough for the relative error to be less than one, that the relative error satisfies

   $$\frac{\sigma_0(N)}{p(K)} \sim \frac{1}{\sqrt{\mathrm{E}[\# \text{ of hits}]}} \; .$$

   Use this to estimate N that would give 10% accuracy when $K = 5$.

   (e) Let $\widehat{M_1}(N)$ be the importance sampled estimator using N i.i.d samples of $v = \mathcal{N}(K,1)$. Evaluate $\mathrm{var}(\widehat{M_1}(N)$ approximately when K is large. Hint: part (1a) is a fact about integrals that can be applied to the variance calculation. Estimate the N needed to estimate $p(5)$ with 10% relative accuracy this way.