Assignment 4, due November 3

Corrections (check the class message board): (none yet.)

1. (*Time change*) This exercise gives a way to turn a Brownian motion into an Ornstein Uhlenbeck process. Suppose W_t is a standard Brownian motion with $\mathrm{E}[\,\Delta W|\mathcal{F}_t] = 0$ and $\mathrm{E}\big[\,\Delta W^2|\mathcal{F}_t\big] = \Delta t$. Here $\Delta W = W_{t+\Delta t} - W_t$, and $\Delta t > 0$. The distribution of W_t has width approximately $\sqrt{t}$, which grows as $t \to \infty$ and goes to zero as $t \to 0$. The Ornstein Uhlenbeck process has a statistical steady state, so its width is not suppose to go to infinity or zero as $t \to \infty$. A function with width $(t^{-1/2} \cdot \sqrt{t})$, such as $Y_t = t^{-1/2} W_t$, should have a width approximately independent of t. A *time change* is a function $t = t(s)$ with $t'(s) = \partial_s t(s) > 0$ when $t > 0$. A time change applied to Y_t gives $X_s = Y_{t(s)}$.

   (a) Define $\Delta Y = Y_{t+\Delta t} - Y_t$ with $\Delta t > 0$. Find formulas for μ_Y and σ_Y^2 so that $\mathrm{E}[\,\Delta Y|\mathcal{F}_t] = \mu_Y \Delta t + \text{(smaller)}$ and $\mathrm{E}\big[\,\Delta Y^2|\mathcal{F}_t\big] = \sigma_Y^2 \Delta t +$ (smaller). By "(smaller)", we mean something that goes to zero faster than Δt, as $\Delta t \to 0$. For example, $\sqrt{t+\Delta t} - \sqrt{t} = \frac{1}{2\sqrt{t}}\Delta t + O(\Delta t^2)$. Here, $f(\Delta t) = O(\Delta t^2)$ means that there is a C and an ϵ so that if $\Delta t \leq \epsilon$, then $f(\Delta t) \leq C\Delta t^2$. This is the "*big O*" notation, and $O(\Delta t^2)$ is read: "on the order of Δt^2". For this problem, (smaller) is the same as $O(\Delta t^2)$. The errors in Taylor series are like this most of the time. This is the mathematicians' idea of order, which refers to scaling rather than size.

   (b) (*Please do this before the next part*) Speculate on the sizes ΔY_t for fixed Δt and large t. Is Y_t changing rapidly or slowly for large t? The rate of change of the Ornstein Uhlenbeck process is roughly constant for all time. If we want X_s to be an Ornstein Uhlenbeck process, should we look for $\Delta t \gg \Delta s$ or $\Delta t \ll \Delta s$?

   (c) Write a formula for σ_X^2 in

$$\begin{aligned}\mathrm{E}\Big[\,(X_{s+\Delta s} - X_s)^2\,|\mathcal{F}_s\Big] &= \mathrm{E}\Big[\,\big(Y_{t(s)+\Delta t(\Delta s)} - Y_{t(s)}\big)^2\,|\mathcal{F}_{t(s)}\Big] \\ &= \Delta s \sigma_X^2 + \text{(smaller)} \\ &= \Delta t \sigma_Y^2 + \text{(smaller)}\end{aligned}$$

   in terms of Δs and $\Delta t = t(s+\Delta s) - t(s) \approx t'(s)\Delta s$. Find and solve the differential equation for $t(s)$ so that σ_X^2 does not depend on s. Does your quantitative solution here agree with the qualitative guess from part (b)?

(d) With the time change from part (c), find a formula for $\mu_X(X_s)$ so that $\mathrm{E}[\, X_{s+\Delta s} - X_s | \mathcal{F}_s] = \mu_X(X_s)\Delta s + \text{(smaller)}$. Does this imply that X_s is an Ornstein Uhlenbeck process?

2. (*Unwinding a conditional expectation. The Brownian bridge construction.*) This exercise is worded in a way that forces you to think through the complexity of our conditional probability and conditional expectation with respect to a $\sigma-$algebra. The result of the calculation is simple and useful – it is the basis for the *Brownian bridge* construction of Brownian motion.

   Suppose W_t is a standard Brownian motion. Suppose $\Delta t > 0$ is a time step size, and $\mathcal{F}_{\Delta t}$ it the $\sigma-$algebra defined by observing the numbers W_{t_k}, where $t_k = k\Delta t$. Let $t_{k+1/2} = (k+\frac{1}{2})\Delta t$. This number is halfway between t_k and t_{k+1}. Let $X = W_{t_{k+1/2}}$ be the value of the Brownian motion at this halfway point. Write a formula for the conditional probability density of X, conditioned on $\mathcal{F}_{\Delta t}$. This formula is Gaussian because many conditioned functions of Brownian motion are Gaussian. It will involve the parameter Δt and observed values W_{t_j} for certain j values. Part of the problem is to figure out which j values enter – think of the Markov property and the Gaussian character of Brownian motion.

3. (*Brownian motion with reflection*) A *reflecting Brownian motion*, with a *reflecting barrier* at $x = a$, is a stochastic process that never crosses a and does not stick to a. For $X_t \neq a$, X_t acts like a Brownian motion. Suppose $X_0 = 0$ and $a > 0$. A reflecting Brownian motion has a probability density, $X_t \sim u(x,t)$, that satisfies the heat equation if $x < a$, and has

$$\int_{-\infty}^{a} u(x,t)\,dx = 1 \;. \qquad (1)$$

   (a) The conservation formula (1) implies a boundary condition that u satisfies at $x = a$. What is this condition? Hint: What must the probability flux be at $x = a$? This boundary condition is called a *reflecting* boundary condition. For `wikipedia` lovers, it is also called a *Neumann* boundary condition.

   (b) Suppose that $v(x)$ is symmetric about the point a, which is the condition $v(a-x) = v(a+x)$ for all x. Show that if if v is a smooth function of x, then v satisfies the boundary condition from part a.

   (c) Adapt the method of images from this week's material to write a formula for the $u(x,t)$ that satisfies the correct initial condition for $X_0 = 0$ and boundary condition at $a > 0$. It is closely related to the formula from class.

   (d) Write a formula for $m_t = \mathrm{E}[\, X_t]$ for reflecting Brownian motion. The *cumulative normal* distribution is $N(z) = \mathrm{P}(\, Z \leq z)$, when $Z \sim \mathcal{N}(0,1)$. Derive a formula for m_t in terms of this and other explicit functions. Verify that m_t is exponentially small for small t. Verify that $m_t \to -\infty$ as $t \to \infty$ and scales as $t^{1/2}$.

(e) It is argued (possibly later in this course, or the book *Stochastic Integrals* by Henry McKean) that a reflecting Brownian motion is kept inside the allowed region $\{x \leq a\}$ by a rightward force at the reflecting boundary. This force is different from zero only when $X_t = a$. The force is just strong enough to prevent $X_t > a$. This picture suggests that the total force is proportional to the total time spent at the boundary. Since only the boundary force has a preferred direction, if $X_0 = 0$, it may be that

$$\mathrm{E}[\, X_t] = \mathrm{E}\left[\int_0^t F_s ds \right] \ ,$$

both sides being negative. Since the force only acts when $X_t = a$, it may be plausible that $\mathrm{E}[\, F_s] = -C\, u(a, s)$. Verify that this picture is true, at least as far as the formula

$$m_t = -C \int_0^t u(x, t)\, dt \ .$$

Find $C > 0$.

4. (*backward equation*) Let X_t be a standard Brownian motion starting from $X_0 = 0$. Let $\tau = \min\{t \ \text{ so that } \ |X_t| = 1\}$. Find the expected hitting time $\mathrm{E}[\,\tau]$. Hint:

   (a) Suppose $V(x, t)$ is a running time reward function and the total reward starting from x at time t is

   $$\int_t^\tau V(X_s, s)\, ds \ .$$

   There the process starts with $X_t = s$, and τ is the first hitting time after t, and $|x| \leq 1$. Define the value function for this to be

   $$f(x, t) = \mathrm{E}_{x,t}\left[\int_t^\tau V(X_s, s)\, ds \right] \ .$$

   Figure out the PDE that f satisfies.

   (b) The case $V(x, t) = 1$ gives the expected hitting time. There is a subtlety here that we need to show $\mathrm{E}[\,\tau] < \infty$. The assignment for a future week will show that there is a $x > 0$ so that $\mathrm{P}_{x,0}(\tau > t) \leq e^{-ct}$. You may assume that $\mathrm{E}[\,\tau] < \infty$ for this exercise.

5. The maximum of a Brownian motion up to time t is

   $$X_t = \max_{0 \leq s \leq t} W_s \ .$$

   Here W_t is Brownian motion and X_t is the “maximum process”. Find the formula for the PDF of X_t. Hint: the statement $X_t \leq a$ is the same as the statement $\tau_a \geq t$. We had a formula for the probability of the latter.

6. (*computing*) This assignment explores the properties of Brownian motion and the Ornstein Uhlenbeck process via simulation. It also introduces you to the slowness of Monte Carlo simulation in general. You have to push the computer pretty hard to get good looking plots. The slowness of `R` does not help.

   This program generates L sample Brownian motion or Ornstein Uhlenbeck paths, all independent, each with time step $\Delta t = T/N$, where T is the end of the time interval and N is the number of time steps, so $t_N = T$. (In the code, we write `dt` for Δt. Let $W_{[0,T]}$ be a path, and $Y = F(w_{[0,T]})$ a function of the path. This assignment just makes histograms of the distributions of various path functionals.

   Download the file `coding.pdf`. It contains guidelines for coding. Please follow them from now on. There are some links on the `resources` page of the course web site that have more material on programming practice. If you watch a good programmer, you will see her or him always following a set of programming guidelines. Ultimately they will save you time in the computing assignments.

   (a) Download the files `Assignment4.R` and `AssignmentStart4.pdf`. If you run the `R` program "out of the box" (exactly as downloaded, all parameters unchanged), you should get a picture that looks like the picture, possibly with different noise. Actually, you need to change one parameter, the name of the directory for the output plot `.pdf` file. This makes a histogram of

   $$F(W_{[0,T]}) = M_T = \max_{0 \leq t \leq T} W_t \ .$$

   The picture `Assignment3.pdf` is a normalized histogram that estimates the probability density, $f(m)$, of the random variable M_T.

   (b) The out-of-the-box picture is not very clear. Try to make it clearer by turning up the computational parameters N and L. Larger N reduces the spurious high value at $m = 0$ and "rounds out" the rest of the picture. We will learn later how it does this. Larger L reduces statistical noise so you can see the curve more clearly. Do this until the run time is more than you have patience for. You will learn two of the prime drawbacks of Monte Carlo: it is slow; it is noisy.

   (c) The code out-of-the-box has partly implemented the study of the distribution of W_T conditional on $W_t \leq B$ for all $t \in [0, T]$. Use the $T = 20$. If you simulate Brownian motion and only count paths that do not touch a *barrier*, you are simulating Brownian motion with *an absorbing boundary* at B. Modify the `R` code to estimate the probability density of W_T conditional on not hitting B before T. You will have to change the parameter `bs` (the *starting bin*) and possibly other things having to do with generating the histogram. Push the computation to get the clearest picture you can with the computer and time constraints you have. Next week we will find a formula for this distribution.

(d) Modify the program to simulate the Ornstein Uhlenbeck process with $\sigma = 2$ and $\gamma = .1$. Start with $X_0 = 2$. On one plot, put the distributions of X_T for $T = 5$ and $T = 10$, both the normalized histograms and the exact formulas for the probability densities. Copy some code from earlier assignments to put multiple plots in the same frame. The exact formulas are from the notes.

(e) Use the explicit formula for the PDF of

$$M_t = \max_{0 \leq s \leq t} X_s$$

and a formula for

$$S_{t,a}(x)dx = \mathrm{P}(\, x \leq X_t \leq x + dx \mid X_s < a \text{ for } 0 \leq s \leq t)$$

Put these graphs in the histograms from part b. Play with parameters to see how good a fit you can get in a reasonable amount of computer time.