Jonathan Goodman, October, 2019

Final exam practice

Information

- The final exam is Monday, December 16 in room 109 from 7:10 to 9pm.
- The exam starts promptly at 7:10, don't be late.
- You are allowed one standard size ($8\frac{1}{2}'' \times 11''$) sheet of paper with any information you like. No other information or electronics are allowed.
- Write all answers in one or more blue books provided. Hand in only the blue books.
- Write your name on each blue book and number them (e.g. 1 of 1, 2 of 3 etc.)
- You will receive 20% credit for question if you write nothing.
- Anything you do write may be counted against you if it is wrong.
- Cross out anything you think is wrong. If you have two answers, the wrong one will count against the right one.
- On multiple choice or true/false questions, give a few words or sentences of explanation. You may lose points even with a correct answer, if it isn't explained.

Practice questions

True/False

1. Let X and Y be random variables with some joint distribution, if X and Y are both random variables, then (X, Y) is a two dimensional Gaussian.
2. If X is a random variable with $\mathrm{E}\left[\, X^2 \right] < \infty$, then $\mathrm{E}\left[\, X^4 \right] < \infty$.
3. If X_t is a diffusion process and $\frac{d}{dt}E[X_t] = 0$, then X_t is a martingale.
4. If $S_{1,t}$ and $S_{2,t}$ are geometric Brownian motions, then $S_t = S_{1,t} + S_{2,t}$ is a geometric Brownian motion.
5. If $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions, then the product rule (Leibniz rule)

$$d[\, f(W_{1,t})\, g(W_{2,t})] = [\, df(W_{1,t})]\; g(W_{2,t})$$

Multiple choice

1. Suppose L is the generator of a diffusion process and $p(x)$ is a PDF Which of the following is true"
   (a) If $g = Lp$, then $g(x) \geq 0$ for all x
   (b) If $g = Lp$, then $\int_{\mathbb{R}^d} g(x)dx = 0$.
   (c) If $g = L^*p$, then $g(x) \geq 0$ for all x.
   (d) If $g = L^*p$, then $\int_{\mathbb{R}^d} g(x)dx = 0$.

2. Suppose $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. Which of the following processes is not a martingale
   (a) $X_t = W_{1,t} + W_{2,t}$
   (b) $X_t = W_{1,t}^3 - 3tW_{1,t}$
   (c) $X_t = W_{1,t}W_{2,t}$
   (d) $X_t = W_{1,t}^2 + W_{2,t}^2$.

Full answer questions

1. Suppose diffusion without drift: $dX_t = b(X_t)dW_t$. Use this formula to show that
$$Y_t = \int_0^t X_s dX_s$$
is a martingale. Use this to evaluate $Y_t = \frac{1}{2}X_t^2 + Q_t$, where Q_t is an integral involving b.

2. If X_t is defined by
$$X_t = \int_0^t s^2 W_s dW_s$$
calculate $\text{var}(X_t)$.

3. Suppose $X_t = W_t^3$ and W_t is standard Brownian motion. Write the SDE that X_t satisfies.

4. Suppose that $X_t = W_{t^2}$ and W_t is standard Brownian motion. Show that X_t is a diffusion and find its infinitesimal mean and variance.

5. Suppose that an $A-$particle does Brownian motion starting from $X_0 = x > 0$, until the first time $X_t = 0$. At that time it is converted into a $B-$particle. Define the probability densities $p(x,t)$ and $q(x,t)$ of for the $A-$particle and the $B-$particle. For example,
$$\Pr\left(\, A - \text{particle } X_t \in [x, x+dx]\right) = p(x,t)dx\ .$$
   (a) Write the PDE and the flux boundary condition satisfied by $p(x,t)$.
   (b) Write a formula for $q(x,t)$.

6. Let X_t satisfy the SDE $dX_t = -\gamma X_t dt + dW_t$. Define

$$f(x,t) = \mathrm{E}_{x,t}\left[\, e^{X_T} \right] \ .$$

   (a) Write the PDE that f satisfies.

   (b) Write the final condition.

   (c) Find a solution of the form $f(x,t) = e^{a(t)x + b(t)}$. Find formulas for $a(t)$ and $b(t)$.

7. Suppose $dX_t = adt + \sigma dW_t$. Let $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$. Find a formula for

$$f(x) = \mathrm{E}\left[\int_0^\infty e^{-rt} H(X_t) dt \mid X_0 = x \right] \ .$$

8. Let X_t be a diffusion process with PDF $p(x,t)$ that satisfies the SDE $dX_t = (1 - X_t)dt + \sigma X_t^2 dW_t$.

   (a) Let $p(x,t)$ be the PDF of X_t. Write the partial differential equation that p satisfies.

   (b) Let $G(y,x,s)$ be the transition density, which means

$$\Pr(\, a \leq X_{t+s} \leq b \mid X_t = y) = \int_a^b G(y,x,s) dx \ .$$

   Write an integral formula for $p(x, t+s)$ in terms of $p(\cdot, t)$ and G.

9. Let W_t be a standard Brownian motion. As in class, define $\Delta t_m = 2^{-m}$ and $t_k = k\Delta t$ for a positive integer m. Define

$$V_t^{(m)} = \sum_{t_k < t} \left| W_{t_{k+1}} - W_{t_k} \right| \ .$$

   (a) Calculate the mean and variance of $V_t^{(m)}$.

   (b) Show that, almost surely,

$$\lim_{m \to \infty} \sqrt{\Delta t_m} V_t^{(m)} = A(t) \ , \quad \text{as } m \to \infty \ .$$

   This includes showing that the limit exists. Find a formula for $A(t)$.

10. Suppose W_t is a standard one dimensional Brownian motion. Define

$$X_t = \int_0^t W_s ds \ .$$

Calculate the mean and variance of X_t^2.

11. Suppose (X_t, Y_t) is a two component diffusion process with infinitesimal mean

$$\begin{aligned} \mathrm{E}[\, X_{t+\Delta t} \mid \mathcal{F}_t] &= X_t + Y_t \Delta t + o(\Delta t) \\ \mathrm{E}[\, Y_{t+\Delta t} \mid \mathcal{F}_t] &= Y_t - X_t \Delta t + o(\Delta t) \end{aligned}$$

and infinitesimal variance/covariance

$$\begin{aligned} \mathrm{var}(\, X_{t+\Delta t} \mid \mathcal{F}_t) &= \left(1 + X_t^2\right) \Delta t + o(\Delta t) \\ \mathrm{cov}(\, X_{t+\Delta t}, Y_{t+\Delta t} \mid \mathcal{F}_t) &= X_t Y_t \Delta t + o(\Delta t) \\ \mathrm{var}(\, Y_{t+\Delta t} \mid \mathcal{F}_t) &= Y_t^2 \Delta t + o(\Delta t) \end{aligned}$$

(a) Write an SDE whose solutions have these infinitesimal means and covariances.

(b) Describe an algorithm for making approximate sample paths for this process.

(c) Suppose we know $X_0 = 0$ and $Y_0 = 1$. Describe an algorithm for estimating $\mathrm{E}\left[\, X_T^2 \right]$.

12. Let N_t be the counting function for the Poisson arrival process with intensity λ. That is, $N_t = \#\left\{T_k < t\right\}$. Let $g = (g_0, g_1, \ldots)$ be a sequence.

(a) Calculate

$$\lim_{t \downarrow 0} \frac{g_{N_t + k} - g_k}{t} \; .$$

(b) Write an expression for Lg, where L is the generator of the Poisson arrival process.

(c) Suppose

$$f(k,t) = \mathrm{E}\left[\frac{1}{N_T + 1} \mid N_t = k \right] \; .$$

Write a family of differential equations that these numbers satisfy.

(d) What extra information besides the differential equations do you need to determine the numbers $f(t,k)$ completely?

13. Suppose $W_t = (W_{1,t}, \ldots, W_{n,t})$ is a standard Brownian motion in n dimensions. Define the *radial process* to be

$$R_t = \left(W_{1,t}^2 + \cdots + W_{n,t}^2 \right)^2 \; .$$

Show that R_t is a Markov process, calculate its infinitesimal mean and variance using Ito's lemma, find the SDE that R satisfies.