Assignment 2, due September 23

Corrections: September 20: Exercise (1g), the max is $t^{-\frac{1}{2}}$ (the original was $t^{\frac{1}{2}}$) Exercise (3a) the formula should be $S = x_0^2 Z^{-2}$ (the original left off the x_0^2 coefficient), Exercise (1b), it should say $g(y) = \frac{1}{Z} e^{-\frac{1}{2}y^2}$ (the original had $e^{-\frac{x^2}{2}}$)

1. (*similarity solution*) We saw in class how to build a solution to the heat equation out of the cumulative normal distribution function. Here is another approach to that.

   (a) Suppose $f(y)$ is some function. Define $u(x,t) = f(t^{-\frac{1}{2}}x)$. The variable $y = x/\sqrt{t}$ is called a *similarity variable*. Find the differential equation $f(y)$ must satisfy in order that u satisfies the heat equation $\partial_t u = \frac{1}{2}\partial_x^2 u$. A solution like this is a *similarity solution*.

   (b) Show that this differential equation is satisfied by a function f with $f'(y) = g(y)$ and $g(y) = \frac{1}{Z} e^{-\frac{1}{2}y^2}$. [A differential equations class explains how to derive this formula for g from $g'(y) = -yg(y)$.]

   (c) Show that the choice $Z = \sqrt{2\pi}$, together with an integration constant in the f equation leads to $f(y) = N(y)$, where N is the cumulative normal distribution function. [This is a derivation of the fact that $u(x,t) = N(t^{-\frac{1}{2}}x)$ satisfies the heat equation.]

   (d) The *Heavyside function* (an example of a *step function*) is defined by $H(x) = 0$ if $x < 0$, $H(0) = \frac{1}{2}$, and $H(x) = 1$ if $x > 0$. Show that $u(x,t) \to H(x)$ as $t \downarrow 0$. [The $\downarrow$ means the limit allowing only values $t > 0$.]

   (e) Let $p(x,0)$ correspond to the uniform probability density in the interval $[0,1]$. Show that $p(x,0) = H(x) - H(x-1)$ and use this to write a formula for $p(x,t)$ in terms of the cumulative normal function N.

   (f) Derive the formula from part (e) using the evolution formula

   $$p(x,t) = \int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\, dy \;.$$

   You might have to use some symmetry properties of $N(y)$, such as $N(-y) = 1 - N(y)$ (why is this true?).

   (g) Use the solution formula to explain the picture of $p(x,t)$ for small t. It consists of a fast transition from 0 to 1 near $x = 0$ and a fast transition back to 0 near $x = 1$. The maximum of $\partial_x p(x,t)$ (with t fixed) is proportional to $t^{-\frac{1}{2}}$, which goes to infinity as $t \downarrow 0$.

2. (*Brownian motion with constant drift*). Let W_t be a Brownian motion with $\mathrm{var}(W_{t_2} - W_{t_1}) = t_2 - t_1$ if $t_2 > t_1$ but the initial density $q(x, 0)$ unspecified. Let $q(\cdot, t)$ be the PDF of W_t. The Brownian motion with constant *drift velocity* a is the process $X_t = W_t + at$. Let $p(\cdot, t)$ be the PDF of X_t.

   (a) Show that $q(x - at, t) = p(x, t)$. *Hint*, the heat equation is not relevant. The fact that W_t is a Brownian motion is not relevant. Use only $X_t = W_t + at$.

   (b) Use part (a) and the fact that q satisfies the heat equation to show that p satisfies a modified equation $\partial_t p = \frac{1}{2}\partial_x^2 p - a\partial_x p$.

   (c) Find an evolution formula for p of the form

   $$p(x, t) = \int_{-\infty}^{\infty} G(x - y, t - t_1) p(y, t_1)\, dy \; .$$

   Write a formula for $G(x, t)$. To derive the formula for G, show that $X_t = X_{t_1} + R$, where R is Gaussian with a specific mean and variance. G is related to the PDF of R.

   (d) Verify by explicit calculation that your G from part (c) satisfies the PDE of part (b). Why is it OK to assume $t_1 = 0$ and $y = 0$ in this verification?

   (e) (*For exercise 4*) Suppose $p(x, t) = e^{\alpha x} e^{\beta t} r(x, t)$. Find α and β so that r satisfies the heat equation. *Hint*, substitute the formula for p in terms of r into the PDE that p satisfies. Find α so that the $\partial_x r$ terms cancel. Then find β so that the r terms cancel. This should leave $\partial_t r = \frac{1}{2}\partial_x^2 r$.

   (f) (*consistency check*) Let $H(x, t) = e^{-\alpha x} e^{-\beta t} G(x, t)$. Show H is what part (e) says it is supposed to be.

3. Consider a Brownian motion without drift ($a = 0$ in exercise 1) that starts from a known place $X_0 = x_0 > 0$ (not random). Let T be the first hitting time for $x = 0$, which is $T = \min\{t \text{ with } X_t = 0\}$. It may be possible that $X_t > 0$ for all $t > 0$. We say $T = \infty$ in that case. Let $u(t)$ be the PDF of T from class 2. We say that an "event" A (An *event* is a set of outcomes.) happens *almost surely* if $\Pr(A) = 1$.

   (a) Let Z be a standard normal random variable, $Z \sim \mathcal{N}(0, 1)$, and define $S = x_0^2 Z^{-2}$. Show by explicit calculation that S and T (the hitting time) have the same PDF.

   (b) Use the result of part (a), or a direct calculation, to show that $T < \infty$ almost surely.

   (c) Show that $\mathrm{E}[T] = \infty$. Write the integral that represents the expectation and show that it diverges. You can do this directly from the known $u(t)$ from class, or you can use the result of part (a).

(d) Justify the statement that a Brownian motion that has not hit the boundary $x = 0$ after a large time t is likely to be far from the boundary. Specifically, show that $\mathrm{E}[X_t \mid t < T] \approx C\sqrt{t}$ for large t. Find C.

4. Consider a Brownian motion with constant drift $a \neq 0$ that starts from a known place $X_0 = x_0$ (not random). Let T be the hitting time for the boundary $x = 0$ as in exercise 2. Let $p(x,t)$ be the PDF of a "surviving particle". For $x > 0$, this is $p(x,t)dx = \Pr(x \leq X_t \leq x + dx \text{ and } t < T)$. Assume that p satisfies the *absorbing boundary condition* similar to the one satisfied by simple Brownian motion: $p(0,t) = 0$.

   (a) Find a formula for $p(x,t)$. *Hint*: use the "exponential weight" transformation from exercise 1 part (e) and then the method of images.

   (b) Use this to find a formula for $u(t)$, which is the PDF of T.

   (c) (*more challenging*) Show that $\mathrm{E}[T] < \infty$ if $a < 0$ and that $\Pr(T = \infty) > 0$ if $a > 0$.

   (d) (*more challenging*) You might think that if $a < 0$, then $\mathrm{E}[T] = x_0/|a|$. If X_t is drifting a distance x_0 at an average speed a, the drift time might be "time = distance/speed". Is this formula true? If not, is it approximately true for large x_0?

Computing exercise

The posted code simulates a discrete time *symmetric random walk* process where Y_k is a location at integer time $k \geq 0$ with $Y_{k+1} = Y_k + \Delta Y$ or $Y_{k+1} = Y_k - \Delta Y$ with probability $p_u = p_d = \frac{1}{2}$. The central limit theorem implies that the distribution of Y_k is approximately Gaussian for large k. More precisely, let X_t be a Brownian motion with $X_0 = Y_0$ Choose Δt so that the increment of X and the increment of Y have the same mean (which is zero for the case without drift, $p_u = p_d$) and the same variance. That is, $\mathrm{var}(X_{t+\Delta t} - X_t) = \mathrm{var}(Y_{k+1} - X_k)$. Define $t_k = k\Delta t$. The central limit theorem says that the distribution of X_{t_k} is a good approximation to the distribution of Y_k for large k.

Moreover, let K be the first k with $Y_k < 0$, which is the random integer discrete hitting time, and let T be the hitting time for the Brownian motion approximation. The *Donsker invariance principle* (more on this later in the course) says that the distribution of K is similar to the distribution of T if many steps are needed to hit the boundary.

Task 1. Download the code `RandomWalkDemo.py`, run it, and check that it produces the same figure `RandomWalkDemoDownload.pdf` that is posted on the web site. The code may take a minute or two to run, depending on your computer. If it takes more time than that, try reducing the parameter m. [This is not a very large computation. The fact that this Python code is slow shows the way you code in Python can make a big difference. This computation probably would be faster in vector Python doing all the paths at once.] The top part of the graph is a plot of the histogram $N_k = \#\{\text{paths with } K = k\}$, as in the computing assignment from Assignment 1. Here, however, the histogram is not normalized to approximate a probability density. The bottom plot has two curves. One is the empirical cumulative probability $\Pr(K < k)$. The other is from the Brownian motion approximation $\Pr(T < t_k)$. Comment on the fat tail (slow decay of the hitting probability density) as $k \to \infty$.

Task 2. The code "out of the box" has a small but visible disagreement between the empirical and theoretical CDF. Modify the parameters of the Y process to make the agreement better. Keep $X_0 = Y_0$ but change Δx. You probably will have to modify `n_max` also. Does increasing the number of paths (parameter m) help? Explain why the change in Δx you used makes the agreement better.

Task 3. If you look at the code that makes the histogram, you will see that it is just a curve, and is not filled in. Yet, out of the box, the plot is solid blue under the curve. Why? What does this say about the convergence of the normalized histogram values to the Brownian motion approximation $u(t_k)$?

Task 4. Modify the up and down probabilities p_u and p_d so that $\mathrm{E}[Y_{k+1} - Y_k] < 0$, but only a little negative. Show how to approximate this process with a Brownian motion with drift. Find a formula for the drift velocity a in terms of p_u, p_d, and Δx. Modify the code you downloaded to use this hitting density. If you choose the parameters carefully, you will see that the fat tails from the Task 1 plot change to thinner tails. The theoretical hitting time density is the one from exercise 4.