Assignment 5, due October 15

Corrections: [none yet]

1. (*Small vs. tiny, from class 4*) Ito's lemma depends on replacing $(\Delta W)^2$ with Δt. The difference is $R = (\Delta W)^2 - \Delta t$. This may not seem "tiny" because R is on the order of Δt, but it is tiny in that a sum of terms like R, if the terms are independent and have mean zero, goes to zero as $\Delta t \to 0$. This is *cancellation*. The sum of the positive terms approximately cancels the sum of the negative terms. Define $R_k = (\Delta W_k)^2 - \Delta t$, with $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. The sum is

$$S = \sum_{t_k < t} R_k \ .$$

   (a) Show that R_k is on the order of Δt in the sense that $\mathrm{E}[|R|] = C\Delta t$. If you have extra time and like doing integrals, show that $C = \sqrt{\frac{2}{\pi e}} \approx .484$.

   (b) Show that $E[R_k^2] = 2\Delta t^2$. Part of this calculation is that $\mathrm{E}_{\mathcal{N}(0,1)}[Z^4] = 3$.

   (c) Show that

$$\mathrm{E}\big[\, S^2 \big] = \sum_{t_k < t} \mathrm{E}\big[\, R_k^2 \big] \ .$$

   The terms on the right are the *diagonal* terms in the double sum

$$\sum_{t_k < t} \sum_{t_j < t} \mathrm{E}[\, R_k R_j] \ .$$

   Why do the *off diagonal* terms (the terms with $j \neq k$) vanish?

   (d) Show that $\mathrm{E}\big[\, S^2 \big] \approx 2t\Delta t$, so $|S|$ is of order $\sqrt{\Delta t}$ as $\Delta t \to 0$.

2. Suppose X_t is a one dimensional Ornstein Uhlenbeck process that satisfies the SDE $dX_t = -\gamma X_t dt + \sigma dW_t$. This exercise leads to understanding the transition probability densities $G(y, x, t)$ for the Ornstein Uhlenbeck process. This is the PDF for transitions from $X_s = x$ to $X_{t+s} = y$. We start with the guess that G is Gaussian, calculate the Gaussian, and verify that this Gaussian works.

   (a) Write the backward equation and use it to calculate the conditional mean of X_{s+t}, which is determined by $\mathrm{E}[X_{t+s} | X_s = y]$. For this, you need to identify T and t (translate the notation we used for backward equations to this context) and the payout function $v(x)$.

(b) Use a similar approach to compute the conditional variance of X_{t+s}. Use the backward equation to calculate $\mathrm{E}[X_{t+s}^2 | X_s = x]$.

(c) Combine this with part (a) to find the PDF of X_{t+s} conditional on $X_s = x$ and assuming that the density is Gaussian. Call this density $G(y, x, t)$, which is a PDF in x in the sense that $X_{t+s} \sim G(y, \cdot, t)$.

(d) Suppose $v(x)$ is a payout function, and define

$$f(y,t) = \mathrm{E}[v(X_T)|X_t = y] = \int_{-\infty}^{\infty} G(y, x, T-t) v(x) dx \; .$$

Show that f satisfies the backward equation in the variables y and t. Do this by putting the derivatives inside the integral and showing that G satisfies this equation for every value of x.

(e) Is G a PDF in the y variable? is $G(\cdot, x, t)$ a PDF for each fixed x (same question)? Hint: Look at G for large t and ask whether that is compatible with $\int G(y, x, t) dy = 1$.

3. Suppose X_j, for $j = 1 \ldots, n$ are n correlated Gaussian prices. Here is a model of correlated prices related what is called CAPM (capital asset pricing model). [Although this model of correlation makes sense, the CAPM overall is the most debunked theory in all of finance.] In this model there is a *market factor* called $Z_0 \sim \mathcal{N}(0,1)$. Asset X_j has coefficient β_j, which is its "beta to the market". This means that $X_j = \beta_j Z_0 +$ *idiosyncratic factor*. [The English word *idiosyncratic* means weird in some individual way. The in finance, idiosyncratic just means individual.] The idiosyncratic factor for X_j is $Z_j \sim \mathcal{N}(0,1)$. All the Z variables are independent. The model is

$$X_j = \beta_j Z_0 + \sigma_j Z_j \; , \quad \text{for} \;\; j = 1, \ldots, n.$$

This represents the n prices using $n+1$ "sources of noise". The class notes say you should never do that, but maybe, ...

(a) Calculate the covariance matrix $C = \mathrm{cov}(X)$ in terms β_j and σ_j.

(b) For $n = 2$ only (unless you really like algebra), find an $n \times n$ matrix B so that $Y = BW$ has the same distribution as X. Here $W \in \mathbb{R}^n$ with $W \sim \mathcal{N}(0, I)$. *Hint:* you can take B to be upper or lower triangular, as in

$$B = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \; .$$

(c) Is the representation from part (b) unique? Is there any other way to represent X using n sources of noise?

(d) Is it possible to find a $Y = BW$ with $W \sim \mathcal{N}(0, I_{m \times m})$ with $m < n$ and B being an $n \times m$ matrix? Is it possible to represent X with fewer than n sources of noise?

4. Suppose A and B have $AA^t = BB^t = C$ where C is non-singular. Show that there is an orthogonal matrix Q (orthogonal means $QQ^t = I$, which is a rotation or reflection in 3d and preserves angles in any dimension) so that $A = BQ$. Show that if the components of a vector Z are independent standard normals, and if $W = QZ$ for an orthogonal matrix Q, then the components of W are independent standard normals.

5. Suppose S_1 and S_2 are two "correlated prices" given by

$$S_{1,t} = e^{\alpha W_{1,t} + \beta W_{2,t}}$$
$$S_{2,t} = e^{\gamma W_{1,t} + \delta W_{2,t}} \ .$$

   (a) Calculate the infinitesimal mean and covariance

$$a(s_1, s_2)dt = \mathrm{E}[\,(dS_1, dS_2) \mid S_{1,t} = s_1, S_{2,t} = s_2]$$
$$\mu(x_1, s_2)dt = \mathrm{cov}[\,(dS_1, dS_2) \mid S_{1,t} = s_1, S_{2,t} = s_2] \ .$$

   (b) Is $S_{1,t}$ a one dimensional Markov process (a diffusion)? Is $(S_{1,t}, S_{2,t})$ a two dimensional diffusion?

Computing exercise

None this week.