Assignment 8, due November 11

Corrections: [none yet]

1. By definition, a (real) *inner product* is a function of two vectors, written $\langle u, v\rangle$, with the properties

$$\begin{array}{lll} \text{(a)} & \langle u,v\rangle = \langle v,u\rangle & \text{(symmetry)} \\ \text{(b)} & \langle u,av\rangle = a\langle u,v\rangle & \text{(homogeniety)} \\ \text{(c)} & \langle u,v_1+v_2\rangle = \langle u,v_1\rangle + \langle u,v_2\rangle & \text{(additivity)} \\ \text{(d)} & \langle u,u\rangle > 0 \text{ unless } u=0 & \text{(positivity)} \;\; . \end{array}$$

Let the vector space be $\mathbb{R}^d$ and C is a symmetric positive definite matrix. Consider the function $\langle u, v\rangle = u^t Cv$.

(a) Show that $\langle u, v\rangle = u^t Cv$ is an inner product by showing that it has these four properties.

(b) Suppose $X \in \mathbb{R}^d$ is a $d-$component random variable with $\mathrm{E}[X] = 0$ and $\mathrm{E}[XX^t] = C$. For any vector $u \in \mathbb{R}^d$, define the scalar random variable $Z_u = u^t X$. Show that $\mathrm{E}[Z_u Z_v] = u^t Cv$. *Hint*: You can do this by calculating with indices, but it may be quicker to use matrix algebra and the trick of writing $v^t X = X^t v$.

(c) Suppose A is a $d \times d$ matrix. Find a formula in terms of A and C for a matrix A^* so that $\langle A^* u, v\rangle = \langle u, Av\rangle$ for all u and v. *Hint:* Show that C^{-1} is a symmetric matrix.

(d) Consider the example

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \;,\;\; A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;,\;\; u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;,\;\; v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;.$$

Calculate A^* and check directly that $\langle A^* u, v\rangle = \langle u, Av\rangle$.

(e) Use the properties (a) - (d) (maybe not all of them) to show that for any inner product, $(A^*)^* = A$.

(f) Check that your formula from part (c) satisfies $(A^*)^* = A$. Do this by matrix algebra with your matrix formula for A^*.

2. Imagine a collection of identically distributed random variables, but not independent, with each distinct pair having the same correlation $\mathrm{corr}(X_i, X_j) = \rho$ for $i \neq j$.

(a) Show that this is possible with $X = (X_1, \ldots, X_d)$ being Gaussian if $0 \le \rho < 1$. Do this by showing that the correlation matrix (ones on the diagonal, ρ in every other entry) is positive definite. Explain why this is enough.

$$C_\rho = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

Hint: Show that a matrix of the form $C = aI + vv^t$ is positive definite if $a > 0$. Find a v and a that gives the specific C_ρ here.

(b) Consider i.i.d. standard normals $Z_0, \ldots Z_d$ ($d+1$ random variables) and define $X_i = aZ_0 + bZ_i$. Find values of a and b that give the desired correlations. The algebra is similar to the algebra of part (a), which is not a coincidence. [Finance people may recognize this as an example of the "market factor" plus "ideosyncratic factors" used by Markowitz.]

(c) Show that C_ρ is not positive definite if $\rho < -\frac{1}{d}$. It is hard (or impossible) to create a bunch of strongly negatively correlated random variables.

(d) For this exercise, take the word "stock" to mean geometric Brownian motion. Let $S_1, \ldots, S_d$ be stocks that satisfy $dS_i = rS_i dt + \sigma S_i dW_i$. Define the joint stock process $S_t \in \mathbb{R}^d$ by $S_t = (S_{1,t}, \ldots, S_{d,t})$. Describe a drift vector $a(s)$ and a noise coefficient matrix $b(s)$ so that each stock separately is a geometric Brownian motion, but

$$\mathrm{corr}(dS_i, dS_j) = \rho\ , \quad \text{if } i \ne j.$$

Hint: One way to do this is to imitate part (b).

(e) Consider the average price process

$$\overline{S}_t = \frac{1}{d} \sum_{i=1}^{d} S_{i,t}\ .$$

Show that $\overline{S}_t$ is not a diffusion process.

(f) Show (informally) that $\overline{S}_t$ is approximately a "stock" (geometric Brownian motion) for large d if $S_{i,0} = 1$ for all i. What is the approximate volatility of $\overline{S}_t$?

3. Consider a single geometric Brownian motion, written as

$$dS_t = \mu S_t dt + \sigma S_t dW_t\ .$$

This exercise involves calculations with forward and backward equations. Do not use Ito's lemma.

(a) Let the value function be $f(s,t)$. Identify the generator L and use it to write the backward equation that f satisfies.

(b) Define a change of variables $g(x,t) = f(e^x, t)$, and calculate the PDE that g satisfies. This involves a change of variables in the backward equation that can be written $s = e^x$, or $x = \log(s)$.

(c) Show that this PDE is the backward equation for a Brownian motion with drift, identify the drift and noise coefficient.

(d) Let $p(s,t)$ be the PDF of S_t, write the adjoint L^* and use it to write the forward equation that p satisfies.

(e) Use the change of variables of part (b) to write the PDE satisfied by $q(x,t) = p(e^x, t)$.

(f) Show that $q(x,t)$ is not the PDF of $X_t = \log(S_t)$. Write a formula for $r(x,t)$ which is the PDF of X_t by including the "jacobian factor" $\frac{ds}{dx}$.

(g) Show that this r satisfies the forward equation corresponding to the backward equation that g satisfies.

4. Consider the one variable Ornstein Uhlenbeck process $dX_t = -X_t dt + dW_t$ with $X_0 = 0$.

   (a) Write the PDF for X_t. *Hint:* It is Gaussian. You need only identify the mean and variance. The mean is easy.

   (b) Turn your answer to part (a) into a formula for $p(x,t)$ in this case. Check by explicit calculation that this satisfies the forward equation.

   (c) Find the value function, $f(x,t)$, for payout $V(x) = x^4$. *Hint* Look for a polynomial solution of the backward equation with the right final conditions.

   (d) Combine your answers to part (a) and part (c) to explicitly evaluate $\mathrm{E}[f(X_t,t)]$ (Here X_t is a certain Gaussian and $f(x,t)$ is a certain polynomial.). The answer will be independent of t, after you get rid of all the algebra mistakes.

5. Suppose X_t is a one dimensional Brownian motion with drift that is confined to the interval $[0,1]$ by boundary forcing as in Assignment 7. That is: $dX_t = aX_t + dW_t + dL_t - dM_t$, where $dL \geq 0$ and $dL = 0$ unless $X_t = 0$, and $dM \geq 0$ and $dM_1 = 0$ unless $X_t = 1$.

   (a) Write an expression for the probability flux, $F(x,t)$.

   (b) Formulate a PDE and boundary conditions and initial conditions that can be used to calculate $p(x,t)$, which is the PDF of X_t.

   (c) Suppose $f(x,t)$ is a value function of the form $f(x,t) = \mathrm{E}_{x,t}[V(X_T)]$.

(d) Formulate a PDE with boundary conditions to be applied at $x = 0$ and $x = 1$, together with final conditions to be applied at $t = T$ that can be used to calculate $f(x,t)$ for $t \leq T$. *Hint:* Use the fact that

$$\mathrm{E}[\,V(X_T)] = \frac{d}{dt}\int_0^1 p(x,t)f(x,t)\,dx$$

is independent of t.

(e) Suppose $p(\cdot,t) \to \pi(\cdot)$ as $t \to \infty$. Assume that the PDF $\pi(x)$ is a steady state for the process. Find a formula for $\pi(x)$.

(f) Suppose $f(x,t) \to g(x)$ as $t \to -\infty$ (or, with fixed t and $T \to \infty$). Show that $g(x)$ is independent of x and give an intuitive reason for this to be true.

(g) Show that the constant of part (f) is equal to

$$\mathrm{E}_\pi[\,V(X)] = \int_0^1 \pi(x)V(x)\,dx\ .$$

Computing exercise

1. Write a finite difference PDE solver for the forward equation for the process of Exercise 5. You can re-use much of the code and ideas from your earlier finite difference PDE solving. Define Δx and Δt and grid points $x_j = j\Delta x$ and solution times $t_k = k\Delta t$. The approximate solution is defined by variables $P_{j,k} \approx p(x_j,t_k)$. For $j = 2,\ldots,n-2$, you can use a simple finite difference method that takes a time step $P_{j,k+1} = \alpha P_{j-1,k} + \beta P_{j,k} + \gamma P_{j+1,k}$. As before, take Δt proportional to Δx^2 and use centered finite difference approximations to $\partial_x p$ and $\partial_x^2 p$ to find α, β, and γ. The update formula for $P_{1,k+1}$ involves the unknown $P_{0,k}$. Find this value by "predicting" $P_{0,k}$ from $P_{1,k}$ using the boundary condition at $x = 0$ from Exercise 5. A similar idea applies for calculating $P_{n-1,k}1$, but now using the boundary condition that applies at $x = 0$. Start with $P_{j,0} = \mathit{const}$ and make plots to show that the solution converges to the steady state probability density from Exercise 5. Plot the computed P and the supposed steady state solution on the same graph to compare. Make a few plots (at least 3, but possibly more) to show how changing computational parameters improves the agreement.

2. Write a code to simulate the process of Exercise 5 and make a histogram of the computed X_T taking many independent sample paths. This histogram should agree with the predicted steady state if T is large enough. Plot the computed histogram and predicted steady state on the same graph, for values of T that show the convergence for large T if there are enough sample trajectories.