Information about the final exam

Instructions and information about the final

1. The exam is real time but online from 7:10 pm to 9pm Monday, December 20.

2. The questions will be posted on the Brightspace site for the class at 7:10pm. You have until 9:05 pm to upload your answers to Brightspace. This probably will be in the form of images of hand-written answers.

3. You must be signed into zoom during the whole exam, with your camera on as a precaution against cheating.

4. The exam is closed-book, closed notes, etc. You may not use any materials written or electronic except the cheat sheet.

5. You are allowed to use a "cheat-sheet", which is one piece of US standard sized paper (8.5in $\times$ 11in) with anything written on it – front and back.

6. Explain all answers with a few words or sentences. Points may be deducted for correct answers that aren't explained. This applies to true/false and multiple choice questions as well as full answer questions.

7. Cross off anything you think is wrong. Points may be deducted for wrong answers even if the right answer also appears.

8. You will get 25% credit for a blank answer. You can lose those points if you give a wrong answer. Cross off anything you think is wrong.

9. Write all answers in the provided exam books. It is desirable but not required to answer the questions in order. Please write clearly and as neatly as possible under exam conditions.

10. The actual exam will have fewer questions than are given below. These are practice questions, not a practice exam.

11. I will be on zoom the whole time. Please chat to me questions or unmute yourself to ask. I may have to chat the answer.

12. Below are a number of sample questions. The actual exam will be shorter, so that it is easily possible to complete it within 110 minutes.

True/False

In each case, say whether the statement is true or false. Explain your answer in a few words or sentences as appropriate. You may lose points even if the answer is correct if the explanation is missing or incorrect.

1. If a random walk process must have Gaussian steps in order to converge to a Brownian motion process.
2. Let τ be the first hitting time for $X_t = 1$ of a Brownian motion starting at $X_0 = 0$. Let Y_t be the stopped process $Y_t = X_t$ if $t < \tau$ and $Y_t = 1$ if $t > \tau$. The expected value of Y_t is zero.

Multiple Choice

In each case, select the one answer you think is most correct. Explain your answer in a few words or sentences as appropriate. You may lose points even if the answer is correct if the explanation is missing or incorrect.

1. Which of the following is not a martingale? W_t is mean zero Brownian motion.

   (a) $X_t = W_t^2 - t$

   (b) $X_t = W_t^4 - 3W_t^2$

   (c) $X_t = tW_t$.

   (d) $X_t = \int_0^t e^s dW_s$

Full answer questions

1. Suppose W_t is a standard Brownian motion and $X_t = a_t W_{t^2}$. Find a_t so that X_t is a standard Brownian motion.
2. Suppose (X, Y) is jointly Gaussian with mean zero and covariance $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$. Find β so that $X - \beta Y$ is independent of Y.
3. Suppose U_n are independent random variables with $U_n = 0$ or $U_n = \pm 1$ and $\Pr(U)n = 0) = \frac{1}{2}$ and $\Pr(U_n = -1) = \Pr|(U_n = 1) = \frac{1}{4}$. Define

   $$S_n = \sum_{k=1}^{n} U_k \ .$$

   Suppose R is a large number. Estimate the distribution of the first hitting time

   $$N = \min \{ n \mid S_n = R \} \ .$$

   Use a Brownian motion scaling to express this, approximately, in terms of a hitting time for Brownian motion.

4. Suppose W_t is standard Brownian motion and

$$X_t = \int_0^t W_s^2 \, dW_s \ .$$

   (a) Suppose $Y \sim \mathcal{N}(0, \sigma^2)$. Show that $\mathrm{E}\big[\, Y^6 \big] = 15\sigma^6$.

   (b) Use the Ito isometry formula to find a formula for $\mathrm{E}\big[\, X_t^2 \big]$. Write it as an integral and calculate the integral.

   (c) Use Ito's lemma to express X_t as $\frac{1}{3}W_t^3 - Z_t$, where

$$Z_t = \int_0^t a_s ds \ .$$

   Find a formula for a_t.

5. Suppose W_t is standard Brownian motion and

$$X_t = \int_0^t W_s \, ds \ .$$

   What is the joint distribution of the two dimensional random variable (W_t, X_t)?

6. Suppose $-a < 0 < b$. Let τ be the first hitting time $W_t = -a$ or $W_t = b$, starting from $W_0 = 0$. Let X_t be the stopped process $X_t = W_{\min(t,\tau)}$.

   (a) Write X_t as an Ito integral with respect to dW_t to show that X_t is a martingale.

   (b) Use the fact that $\Pr(\tau > t) \to 0$ as $t \to \infty$ and the martingale property to identify the limits of $\Pr(X_t = -a)$ and $\Pr(X_t = b)$ as $t \to \infty$.

7. The *total variation* of a continuous function f_t is

$$\mathrm{TV}(f, T) = \lim_{\Delta t \to 0} \sum_{t_k < T} \left| f_{t_{k+1}} - f_{t_k} \right| \ .$$

   This uses our usual convention that $t_k = k\Delta t$. Show that the total variation of Brownian motion is infinite as long as $T > 0$. More precisely, show that for small Δt

$$\sum_{t_k < T} \left| W_{t_{k+1}} - W_{t_k} \right| \approx \frac{C_T}{\sqrt{\Delta t}} \ .$$

   Identify the constant C_T and explain the role of the law of large numbers or central limit theorem in this approximate formula.

8. Find functions $A(t)$ and $B(t)$ so that $f(x,t) = x^4 + A(t)x^2 + B(t)$ satisfies the backward equation

$$f(x,t) = \mathrm{E}\big[\, X_T^4 \mid X_t = x \big] \ . \ \text{where } X_t \text{ is standard Brownian motion}$$

9. For X_t beint ordinary Brownian motion, let $f(x,t)$ be the value function for the final time payout $V(X_t) = (X_T)_+$. Write an integral formula in terms of the Green's function G (Week 3). Calculate the integral. The formula involves an exponential part and a part involving the cumulative normal $N(x)$.

10. Suppose there is a random payout at time T, so that you get $V(X_T)$ with probability p and zero with probability $1-p$. Let $f(x,t)$ be the expected value of this random payout if $X_t = x$ and X_t is Brownian motion. Find the backward equation and final condition that f satisfies.