# Assignment 2

**Correction**. Formula (4) in Exercise 3 is corrected to replace $\sqrt{2\pi}$ with the correct $\sqrt{2\pi t}$.

1. This exercise is a somewhat informal derivation of the *Fick's law* or the *Fourier law* for the *probability flux* for a diffusion process

$$dX_t = a(X_t)dt + b(X_t)dW_t \ .$$

Suppose $p(\cdot, t)$ is the PDF of $X_t$ and $C(x,t)$ is the CDF (cumulative distribution function)

$$C(x,t) = \Pr(X_t \leq x) = \int_{-\infty}^{x} p(y,t)\, dy \ .$$

The *probability flux* (also called probability *current*) is defined by

$$\partial_t C(x,t) = -F(x,t) \ .$$

This is explained more in the notes and will be covered in Class 3. The Fourier/Fick law is

$$F(x,t) = a(x)\, p(x,t) - \frac{1}{2}\partial_x \left[ b^2(x)\, p(x,t) \right] \ . \tag{1}$$

The derivation uses

$$\partial_t C(x,t) = \lim_{\Delta t \downarrow 0} \frac{C(x,t+\Delta t) - C(x,t)}{\Delta t}$$

The "downarrow" symbol $\downarrow$ in the limit indicates that $\Delta t > 0$ and goes down to zero, while $\Delta t < 0$ is not considered. There are two ways $C(x,t)$ can change as $t$ increases. A particle from the right of $x$ at time $t$ can cross to the left at time $t + \Delta t$ or a particle can cross from left to right. The probabilities of these events are

$$\begin{aligned} P_U &= \Pr\left(\, X_t < x \text{ and } X_{t+\Delta t} \geq x \,\right) \\ P_D &= \Pr\left(\, X_t > x \text{ and } X_{t+\Delta t} \leq x \,\right) \ . \end{aligned}$$

We will use three approximations, which are intuitive and correct but not proven in this Exercise. The first is

$$p(y,t) \approx p(x,t) + [\partial_x p(x,t)](y-x) \ .$$

This should be accurate for the $X_t = y$ values relevant for $P_U$ and $P_D$ because a particle that crosses $x$ in time $\Delta t$ must start close to $x$ at time

$t$. The second approximation is the Euler Maruyama time step approximation that uses the standard normal $Z \sim \mathcal{N}(0,1)$:

$$X_{t+\Delta t} \approx X_t + a(X_t)\Delta t + \sqrt{\Delta t}\, b(X_t)\, Z \; . \tag{2}$$

The third approximation is the "method of fractional steps", which is the fact (conjecture?) that (1) is true if it is true when $a = 0$ and also when $b = 0$. This just turns one big calculation involving both $a$ and $b$ into two simpler calculations.

The verification when $b = 0$ is simple. If $a > 0$ then $P_D = 0$. The verification when $a = 0$ is the main point. For this, it may help to use the *cumulative normal* distribution function

$$N(x) = \Pr(Z < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz \; .$$

Both $P_U$ and $P_D$ can be expressed as integrals involving $N$, if we use the Euler approximation (2). This expresses $P_U$ as a double integral. Reversing the order of integration gives a formula involving $\partial_x p(x,t)$ and $\mathrm{var}(X_{t+\Delta t} | X_t = y)$.

2. An *Ornstein Uhlenbeck* process is any diffusion process whose SDE is linear in the drift and constant in the noise. In one dimension, that is

$$dX_t = -\gamma X_t\, dt + \sigma dW_t \; . \tag{3}$$

Parts of this exercise require $\gamma > 0$, but $\gamma < 0$ come up in other applications. Assume that $X_0$ has a PDF $p(x,0)$. To understand dynamics, make the Euler approximation involving the standard normal $Z$:

$$X_{t+\Delta t} \approx X_t - \gamma X_t \Delta t + \sigma\sqrt{\Delta t}\, Z \; .$$

When computing a derivative, take the difference quotient with $\Delta t > 0$, use the Euler approximation, and take $\Delta t$ to zero.

(a) The mean at time $t$ is

$$m(t) = \mathrm{E}[\, X_t \,] \; .$$

Show that

$$\frac{d}{dt} m(t) = -\gamma m(t) \; .$$

Show that $m(t) \to 0$ as $t \to \infty$ as long as $|m(0)| < \infty$.

(b) The mean square and variance are

$$S(t) = \mathrm{E}\big[\, X_t^2 \,\big] \;\; , \;\; V(t) = \mathrm{var}[\, X_t \,] \; .$$

Calculate $\frac{d}{dt}S$ and $\frac{d}{dt}V$ and use these to find

$$V_\infty = \lim_{t\to\infty} \mathrm{var}[\, X_t \,] \; .$$

*Remark.* The formula for $\frac{d}{dt}V$ has a positive term that depends on $\sigma$ and represents the increase in uncertainty coming from noise. It also has a negative term that depends on $\gamma$ that represents the inward "push" of the drift term. The formula for $V_\infty$ represents a balance of these forces. It should show that $V_\infty$ increases when $\sigma$ increases and decreases when $\gamma$ increases. *Note.* You do not need to solve the differential equations that $S$ and $V$ satisfiy. Setting $\frac{d}{dt}S = 0$ and using $m_\infty = 0$ gives a simple algebraic equation for $S_\infty$. If $S(t)$ has a limit as $t \to \infty$, then the derivative goes to zero. [Not always, but in this case].

(c) Find the differential equation and limiting value as $t \to \infty$ of

$$Q(t) = \mathrm{E}\left[\, X_t^4 \,\right] \; .$$

Take $m(t) = 0$ in order to simplify the computations. *Note.* You do not need to solve the differential equation that $Q$ satisfies. Setting $\frac{d}{dt}Q = 0$ gives a simple algebraic equation for $Q_\infty$.

(d) Assume that $X_0 = 0$ so that $p(x,0) = \delta(x)$ (the "delta function" is described below). Assume that $X_t$ is Gaussian and use the results of parts (a) and (b) to find a formula for $p(x,t)$. Verify by direct calculation that this $p(x,t)$ satisfies the forward equation for (3). *About the delta function.* The *delta function* is an idealized function, or improper function (also called *distribution*) that has $\delta(x) = 0$ if $x \neq 0$ but $\int \delta(x)\, dx = 1$. This is a "unit mass" (or unit of area) all at the point $x = 0$. A Gaussian with mean zero and variance $\epsilon$ has density

$$p_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \; .$$

This converges to $\delta(x)$ as $\epsilon \to 0$ in the sense that if $f$ is any bounded and continuous function, then

$$\int_{-\infty}^{\infty} f(x)\, p_\epsilon(x)\, dx \;\to\; f(0)\;, \quad \text{as } \epsilon \to 0\;.$$

If $Y$ is a random variable with PDF $q(y)$, then saying $Y = a$ is the same as saying $q$ is a "point mass" or "delta mass" at $y = a$, which is $q(y) = \delta(x - a)$.

(e) Show that if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$\mathrm{E}\left[\, X^4 \,\right] = 3\sigma^4 \; .$$

Show that the quantities $V_\infty$ and $Q_\infty$ are consistent with this, using the fact that the Ornstein Uhlenbeck PDF is Gaussian. *Hint.* Integrate by parts using

$$\int_{-\infty}^{\infty} z^4 e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{\infty} z^3 \left( e^{-\frac{z^2}{2}}\, z \right) dz$$

3. Let $X_t$ be a "Brownian motion started at $X_0 = a$". We sometimes understand "Brownian motion" to mean starting at $W_0 = 0$, so you might be reluctant to call $X_t$ Brownian motion. If you want, take $X_t = W_t + a$, or say $X_0 = a$ and $dX_t = dW_t$. The first *hitting time* at $x = 0$ is

$$\tau = \min \{ \, t \mid X_t = 0 \, \} \ .$$

We say that $\tau = \infty$ if $X_t > 0$ for all $t$. There are paths that do this, but they might be unlikely. The PDF of $\tau < \infty$ is

$$q_a(t) = \frac{1}{\sqrt{2\pi t}} \, \frac{2a}{t} \, e^{-\frac{a^2}{2t}} \ . \tag{4}$$

This exercise asks you to verify this directly from simulations of hitting times. You can simulate a hitting time using the usual Euler approximation $X_0 = a$ and

$$X_{t_{n+1}} = X_{t_n} + \sqrt{\Delta t} \, Z_n \ .$$

This Euler formula is exact for Brownian motion. You define

$$\tau_{\Delta t} = \min_{t_n} \{ \, t_n \mid X_{t_n} < 0 \, \} \ .$$

You should stop the simulation at $t_n = t_{\max}$ because otherwise the expected computer time is infinite (it, almost literally, takes forever). Exercise 4(c) explains this. You estimate $q(a)$ by doing many simulations and making a histogram. Make a figure that plots this histogram for several values of $\Delta t$ and also contains the curve (4). Use enough paths so that the noise in histograms is hard or impossible to see in the plots. Choose the histogram bins to that each bin contains an integer number of time steps. This will constrain the values of $\Delta t$ you use. Normalize the histogram so that it estimates the PDF, as was done in Assignment 1. Choose values of $\Delta t$ to show that the histogram for the largest $\Delta t$ is clearly visible in the figure but the convergence as $\Delta t \to 0$ also is clear. Hand in one or more figures (not more than 3), a printout of your code, and some comments about the results and the running times.

4. This exercise gives some insight into the hitting time PDF (4).

   (a) Let $Z$ be a standard normal $Z \sim \mathcal{N}(0,1)$ and take $T = \frac{a^2}{Z^2}$. Show that the PDF of $T$ is also given by (4).= *Remark.* It is easier to simulate random hitting times using this trick than the method of Exercise 3, but the trick here does not apply to general diffusion processes.

   (b) Show that $\tau < \infty$ with probability 1. An event with probability 1 is said to happen *almost surely.* In other words, show that

   $$\int_0^\infty q_a(t) \, dt = 1 \ .$$

   *Hint.* Use (a).

   (c) Show that $\mathrm{E}[\tau] = \infty$. *Hint.* This can be done directly or using (a).