Assignment 3

Correction. Exercise (1b) has been corrected to make $p(x,T)$ a differentiable function of x for all x. In this version, $\partial_x p(\pm\pi, T) = 0$ and $p(x,T)$ is given by a simple formula for all other values of x. The notation of Exercise (1c) has been clarified. Exercise 3 has been clarified to say that you should use parameters and initial conditions that lead to a steady state probability density.

1. In each case, explain why there is no PDF $p_0(x)$ so that if X_t is Brownian motion with $X_0 \sim p_0$ then $X_T \sim p(\cdot, T)$. You may use the forward uniqueness theorem that if $p_0(x) \neq q_0(x)$ then $p(\cdot,t) \neq q(\cdot,t)$ for $t > 0$ (here, q is the solution of the PDE with initial data q_0).

   (a) $T = 2$, and $p(x,T) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. *Hint.* variance.

   (b) $T = .5$ and

   $$p(x,T) = \frac{1}{2\pi} \begin{cases} 1+\cos(x) & \text{if } |x| \leq \pi \\ 0 & \text{if } |x| > \pi \end{cases} .$$

   The $\frac{1}{2\pi}$ factor makes $p(\cdot,T)$ a proper PDF. *Hint.* smoothness.

   (c) $T = .5$ and $p(x,T) = \frac{11}{10}\mathcal{N}(0,100) - \frac{1}{10}\mathcal{N}(0,1)$. The notation $\mathcal{N}(\mu,\sigma^2)$ refers to a function of x given by $q(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. This is the Gaussian density with mean μ and variance σ^2 *Hint.* sign.

2. A *reflecting boundary* is a boundary that does not allow a particle to pass through and does not absorb the particle. The forward equation for a reflecting process is the same as the forward equation with no constraints. The difference is the *boundary condition* that is applied at the reflecting boundary. In one dimension, that boundary condition is that the probability flux is zero at the reflecting boundary.

   (a) Suppose the process is Brownian motion starting from a point $x_0 > 0$ and the reflecting boundary is at $x = 0$. Use the method of images to write a formula for $p(x,t)$, the PDF of X_t, with $t > 0$ and $x \geq 0$. *Hint.* Show that the boundary condition automatically satisfied if p is extended to $x < 0$ by making it an even function of x: $p(-x,t) = p(x,t)$.

   (b) Find an approximate formula for $m_t = \mathrm{E}[X_t]$ for this reflecting Brownian motion process that is valid for large t. The answer should take the form

   $$m_t \approx \text{something that goes to } \infty \text{ as } t \to \infty .$$

This "something" should have a definite power of t and constant. The constant involves $\frac{1}{\sqrt{\pi}}$. It would be best (but not absolutely required) to show that $|m_t -$ "something"$|$ is bounded as $t \to \infty$. *Hint.* The distribution of X_t does not change much if you replace x_0 with 0. Why not (give a mathematical explanation that uses the formula for $p(x,t)$ when t is large.)?

(c) Consider reflecting Brownian motion with a constant drift rate $a < 0$. What is the zero probability flux boundary condition? Find a time independent function $p_\infty(x)$ that satisfies the forward equation with $\partial_t p_{|infty}(x) = 0$ and the reflecting boundary condition. This is an *steady state* or *equilibrium* probability distribution in the sense that if $X_0 \sim p_\infty$ then $X_t \sim p_\infty$ slso for $t > 0$.

(d) Give an intuitive explanation why you do not expect a steady state probability distribution to exist if $a = 0$ or $a > 0$.

3. Write a code to simulate reflecting Brownian motion. Enforce the reflecting boundary condition by $X_{t_{n+1}} = \left| X_n + \sqrt{\Delta t} \ \cdots \right|$. Make histograms of $p(\cdot, t)$ for various t values. Choose $x_0 > 0$ and $a < 0$. Choose the t values to demonstrate the following qualitative features

   - For small t, when X_t does not yet "feel" the boundary, $p(\cdot, t)$ is approximately Gaussian. Plot that Gaussian over the histogram to demonstrate quantitative agreement.
   - For intermediate t, when X_t has encountered the boundary but has not reached equilibrium, $p(x,t)$ is not Gaussian but has not yet achieved the equilibrium distribution.
   - $p(\cdot, t) \to p_\infty(\cdot)$ as $t \to \infty$. Plot p_∞ and the histogram in the same frame to demonstrate quantitative agreement.

4. Do a simulation of the reflecting Brownian motion with drift using the same x_0 and a that you used in Exercise 3. For this exercise, use only one simulation but run it for a very long time. Make a histogram of the values of X_{t_n}, for values t_n starting at a time (known from Exercise 3) when the distribution is close to equilibrium. This illustrates a fundamental property of random processes that have steady state distributions: a single path samples the steady state distribution if the path is long enough. This is often called *ergodic* behavior. You will see that it takes a very long path to get a clean histogram because the numbers X_{t_n} are not independent.

5. Define $f(x,t)$ to be the value function

$$f(x,t) = \mathrm{E}[\, V(X_T) \mid X_t = x \,]$$

   In each case, calculate $f(x,t)$ directly using a Gaussian expectation of V with the appropriate Gaussian, then check that the f you get satisfies the backward equation with the final condition $f(x,T) = V(x)$.

(a) Take X_t to be standard Brownian motion with no drift and $V(x) = x^2$.

(b) Take X_t to be standard Brownian motion with no drift and $V(x) = x^4$),

(c) Take X_t to be Brownian motion $dX = adt + \sigma dW_t$ and $V(x) = x^2$.