Assignment 4

1. Let X_t be a standard Brownian motion. In the usual notation, $t_k = k\Delta t$ for $k = 0, 1, \cdots$. The Δt approximation to the quadratic variation is

$$[X]_T^{(\Delta t)} = \sum_{t_k < T} \left(X_{t_{t+1}} - X_{t_k} \right)^2 \ .$$

Show that $[X]_T^{(\Delta t)} \to T$ as $\Delta t \to 0$ in the sense that its mean is close to T

$$\left| \mathrm{E}\Big[\, [X]_T^{(\Delta t)} \Big] - T \right| \le \Delta t \ .$$

and its variance goes to zero. More precisely, find the exponent p and the prefactor A so that

$$\mathrm{var}\Big(\, [X]_T^{(\Delta t)} \Big) \approx A \Delta t^p \ , \quad \text{as } \Delta t \to 0 \ .$$

2. Let S_t be a geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$. The quadratic variation formula for this is

$$[S]_T = \sigma^2 \int_0^T S_t^2 \, dt \ .$$

Make the standard approximation to the quadratic variation

$$[S]_T^{(\Delta t)} = \sum_{t_k < T} \left(S_{t_{t+1}} - S_{t_k} \right)^2 \ .$$

Choose a simulation step size h that is much smaller than Δt and use this to generate sample paths S_t and to estimate the regular integral

$$\int_0^T f_t \, dt \approx h \sum_{jh < T} f_{jh} \ .$$

In general, *tracking error* can mean the difference between a formula from Ito calculus and its approximation with finite Δt. Make histograms of the tracking error of the approximate quadratic variation $[S]_T^{(\Delta t)}$ to demonstrate, computationally, that the tracking error goes to zero as $\Delta t \to 0$. Choose several values of Δt and try to see the power p in Exercise 1 reflected in the scale of the tracking error here as a function of Δt.
Note. Many details have purposefully been left out. Please choose details and parameters to make the results interesting, the problem non-trivial, and the computation time practical. This will take some experimentation.

Choose T, μ and σ so that you can see both the exponential growth of the mean and significant statistical variation in S_T. Does it matter how you choose S_0? Choose a Δt sequence that ranges from "large" (not many time steps needed to reach T) to considerably smaller in order that the trend of tracking error is clear. Choose h small enough so the sample paths are "essentially exact", which means that the results are "converged" in the sense that they would not change visibly if you made h smaller.

3. Solve the backward equation for Brownian motion and final payout $V(x) = e^{-\frac{1}{2\sigma^2}x^2}$. You can do this by systematic guessing (then checking that it's right), or by using the fundamental solution, or another way. Let the value function be $f(x,t)$. Why does $f(x,0) \to 0$ as $T \to \infty$ ($T =$ final payout time), or for fixed T as $\sigma \to 0$? *Hint.* Finance people say an option is "in the money" (or, "finishes in the money") if the payout is significantly larger than zero. What are the odds of finishing in the money in this example?

4. Consider a one component diffusion $dX_t = a(X_t)dt + b(X_t)dW_t$. Choose $\Delta x > 0$ and $\Delta t = \lambda \Delta x^2$ and use these to create a random walk approximation to the SDE. The spatial grid points are $x_j = j\Delta x$ and the temporal grid points are $t_k = k\Delta t$. The approximating random walk has $Y_k \approx X_{t_k}$, but $Y_k = x_j$ for some j. In each time step the random walk can "hop" to the left, which is $Y_{k+1} = Y_k - \Delta x = x_{j-1}$, or it can "stay put", which is $Y_{k+1} = Y_k$, or it can hop to the right, which is $Y_{k+1} = Y_k + \Delta x = x_{j+1}$. The hopping probabilities are

$$Y_{k+1} = \begin{cases} Y_k - \Delta x & \text{with probability } p_j \\ Y_k & \text{with probability } q_j \\ Y_k + \Delta x & \text{with probability } r_j \end{cases}$$

The hop at time t_k is $\Delta Y_k = Y_{k+1} - Y_k$, which is $\pm\Delta x$ or zero.

   (a) Find formula for p_j, q_j, and r_j so that $\mathrm{E}[\Delta Y_t] = a(x_j)\Delta t$ and $\mathrm{E}\big[\,(\Delta Y_k)^2\big] = b(x_j)^2\Delta t$. Why do you have to assume λ is not too large?

   (b) Suppose there is a payout at time $T = t_n$ and there is a discrete value function with values g_{jk} defined by

$$g_{jk} = \mathrm{E}[\,V(Y_n) \mid Y_k = x_j]\ .$$

   Find a backward equation for the numbers g_{jk} of the form

$$g_{jk} = (*)g_{j-1,k+1} + (**)g_{j,k+1} + (***)g_{j+1,k+1}\ . \qquad (1)$$

   Identify the coefficients, $(*)$, etc., in terms of the hopping probabilities p_j, q_j, and r_j.

(c) Suppose there are absorbing boundaries at $x_{\min} = 0$ and $x_{\max} = R\Delta x > 0$, and the payout is zero if $Y_k = x_{\min}$ or $Y_k = x_{\max}$ for any $k = 0, 1, \cdots, n$. This means that at any time t_k there are $R - 1$ unknowns g_{jk} for $j = 1, \cdots, R-1$ while $g_{0,k} = 0$ and $g_{R,k} = 0$. Write a code to apply the finite difference formulas (1) and compute the approximate value function values $g_{j,0}$. Make a plot as a function of x (not j, which depends on the numerical parameter Δx). Show that the computed answer converges as $\Delta x \to 0$. Take the Ornstein Uhlenbeck SDE $dX = -X dt + dW$, final time $T = 1$, payout $V(x) = x^2$, and $x_{\max} = 2$. Figure out what λ you need.
Notice that the solution goes to zero at the absorbing boundaries.