Practice for the Final Exam

Information and rules for the final

- The final exam is 7:10 to 9pm on Monday, December 19.
- No electronic devices may be visible during the exam.
- The exam is "closed book". No materials are allowed except for one "cheat sheet", which is one standard piece of paper that you may prepare in advance with any information you want.
- Please explain all answers with a few words or sentences. You may lose points for a formula or answer with no explanation.
- You will get 25% credit for any question left blank. Points may be deducted for wrong answers, which can bring the credit to zero.
- Please cross out any answers you think are wrong. You may lose points for wrong answers even if you also give the right answer.
- Write answers in an exam book ("blue book").

Preliminary, more questions to come

True/False

In each case, say whether the statement is true or false and give few words or sentences to justify your answer.

1. If (X, Y) is a two component random variable with PDF $p(x, y)$, and if X and Y are both Gaussian, then (X, Y) is a two component Gaussian.
2. If $g(x)$ is continuous and has $|g(x)| \leq 1$, then there is a continuous function $f(x, t)$ defined for $t \geq 0$ that satisfies $\partial_t f + \frac{1}{2}\partial_x^2 f = 0$ and $f(x, 0) = g(x)$.
3. If $f(x, t)$ satisfies $\partial_t f + x\partial_x f + \partial_x^2 f = 0$ and $|f(x, T)| \leq 3$ for all x, then $|f(x, t)| \leq 3$ for all x if $t < T$.
4. If X_t satisfies the SDE $dX_t = a(X_t)dt + b(X_t)dW_t$ then X_t is an Ito process.

Multiple Choice

In each case, select the correct answer and justify your choice with a few words or sentences.

1. Let $f(x,t)$ be the value function defined for $t < T$ by

$$f(x,t) = \mathrm{E}\left[\int_t^T V(X_s)\, ds \mid X_t = x \right] .$$

   Which condition is true about f?

   (a) $f(x,T) = V(x)$
   (b) $f(x,t) = 0$
   (c) $f(x,T) = 1$
   (d) $f(0,t) = 0$

2. In which case is X_t not a Markov process? Assume W_t, $W_{1,t}$ and $W_{2,t}$ are independent Brownian motion processes.

   (a) $X_t = \displaystyle\int_0^t t^2 dW_t$
   (b) $dX_t = X_t dt + dW_t$.
   (c) $dX_t = Y_t dt + dW_{1,t}$, $dY_t = X_t + dW_{2t}$.
   (d) $dX_t = X_t + dW_{1,t} + dW_{2,t}$, $dY_t = Y_t + 2dW_{1,t}$.

3. Let $\mathcal{L}$ be the generator of a diffusion process and $g = \mathcal{L}f$. Which of these is true?

   (a) If $f(0) = 0$ then $g(0) = 0$
   (b) If $f(x) = 1$ for all x then $g(x) = 1$ for all x.
   (c) If $f(x) = 1$ for all x then $g(x) = 0$ for all x.
   (d) If $f(x) \geq 0$ for all x then $g(x) \geq 0$ for all x.

Full answer

The difficulty of these problems is different from problem to problem.

1. Suppose $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions, and (X_t, Y_t) is a two component process (X_t, Y_t) that satisfies

$$dX_t = -X_t dt + X_t dW_{1,t} + W_{2,t}$$
$$dY_t = Y_t dW_{1,t} + X_t dW_{2,t} \ .$$

   Write an SDE the could be used to generate sample paths of X_t alone using a single Brownian motion process W_t.

2. Write an SDE system to describe a pair (X_t, Y_t) so that

- X_t is a geometric Brownian motion with infinitesimal growth rate $\mu_t = Y_t$ and volatility σ,
- Y_t is an Ornstein Uhlenbeck mean reverting process with mean reversion rate γ and infinitesimal variance $\mathrm{E}[(dY)^2] = 1$,
- dX_t and dY_t have correlation coefficient $\rho = .5$.

3. Consider the stochastic volatility model

$$\begin{aligned} dS_t &= \mu S_t dt + Y_t S_t dW_{1,t} \\ dY_t &= -\gamma(Y - \overline{\sigma})dt + \rho dW_{2,t} \end{aligned}$$

Suppose there is a final time payout $Z = V(S_T)$. How would you make a histogram that approximates the PDF of Z?

4. Let W_t be a standard Brownian motion and suppose $\mathcal{F}$ is the $\sigma-$algebra generated by $W_{[0,1]}$ and $W[3,4]$. That means that $\mathcal{F}$ "knows" the path for t between zero and 1 and for t between 3 and 4. Describe the random variable $\mathrm{E}[\, W_2 \mid \mathcal{F}\,]$.

5. Let W_t be a standard Brownian motion. Describe the distribution of W_2 conditional on $W_1 = a$ and $W_3 = b$.

6. Let X_t satisfy $dX_t = X_t dt + dW_t$ with $X_0 = 1$. What is the PDF of X_1?

7. Suppose $dS_t = \mu S_t dt + \sigma S_t dW_t$. Suppose there is a payout $V(s) = S^2$. Find a formula for the value function

$$f(s,t) = \mathrm{E}[\, V(S_T) \mid S_t = s] \ .$$

8. Suppose $dS_t = \mu S_t dt + \sigma S_t dW_t$, with $S_0 = 1$. Calculate $d(S_t)^2$ and use that to calculate

$$\mathrm{E}\left[\, S_t^2 \right] \ .$$

9. Suppose $dS_t = \mu S_t dt + \sigma S_t dW_t$. For a Δt, define times $t_k = k\Delta t$ and realized returns

$$R_k = \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} \ .$$

Find the distribution of the short time *realized variance*

$$\lim_{\Delta t \to 0} \sum_{t_k < t} R_k^2 \ .$$