Introduction to Mathematical Modeling

Jonathan Goodman
Courant Institute of Mathematical Sciences, NYU

Class 2

We will continue our modeling warm up by reviewing ordinary differential equations and linear algebra, and by doing some more approximations along the way. The physical context for this discussion will be oscillators, simple physical systems that oscillate. We will be able to estimate the way oscillators lose energy (wind down) due to friction. We will also study the phenomenon of "resonance", when oscillators with (nearly) identical frequencies interact.

Newton's laws and the harmonic oscillator

First, the physics. Newton's law of motion (a mathematical formulation of his three laws of motion) is

$$f = ma \; . \tag{1}$$

Here f is the force acting on the object from outside it, m is the mass of the object, and a is its acceleration. If $x(t)$ is the position of the object at time t, then $v(t) = \frac{dx}{dt} = \dot{x}(t)$ is its velocity, and $a(t) = \frac{dv}{dt} = \dot{v}(t) = \frac{d^2x}{dt^2} = \ddot{x}(t)$ is the acceleration. One consequence of (1) is that there is no acceleration if there is no force. Without acceleration, an object moves in a straight line at constant speed. This is the content of Newton's first law of motion: "Every body continues in its state of rest, or of uniform motion in a right (*i.e. straight*) line unless it is compelled to change that state by forces impressed upon it."[1] This may not have been completely intuitive in Newton's time, as air hockey had not yet been introduced. The second law says what happens when $f \neq 0$: "The change of motion is proportional to the motive force impressed and is made inthe direction of the right line in which that force is impressed."

One interesting consequence of Newton's law (1) relates to rockets. Most of the mass of a rocket at launch time is its fuel. The rocket engines must generate enough force to lift the full fuel tanks off the ground. If the force generated by the engines remains constant, the acceleration will increase with time as the fuel is depleted; if ma is constant, then a goes up as m goes down. This makes

[1] Sir Isaac Newton, *Mathematical Principles of Natural Philosophy* (better known by its Latin title: *Philosophiae Naturalis Principia Mathematica*, which means the same thing, or simply as *Principia*, 1686, translated into English from the original Latin by Andrew Motte in 1729, modern edition, University of California Press, page 13.

the ride for astronauts increasingly uncomfortable, with the greatest discomfort right before the fuel runs out and the engines go off.

A physical model of an oscillator is a mass on the end of a spring.[2] We suppose that the mass moves back and forth on a "right" line while connected via a light but strong spring to a rigid wall. We denote the displacement of the mass from its resting position by x. If the mass is displaced from its resting position, the spring pushes it back. The force is a *restorign force*, f is negative when x is positive and vice versa. For simplicity, we assume a *linear* spring. This means that the force applied by the spring on the mass is proportional to the displacement:

$$f = -kx \; . \tag{2}$$

Many oscillators, such as the heart, are nonlinear. The spring-mass system is reasonably close to linear if x is small enough. Combining (1) and (2) gives the differential equation for the motion of the oscillator:

$$m\ddot{x} = -kx \quad . \tag{3}$$

If we put the m on the other side, we get

$$\ddot{x} = -\frac{k}{m}x \quad . \tag{4}$$

Let us try to guess the form of the solution and the shape of the graph of $x(t)$ under the hypotheses that at time $t = 0$, $x = 0$ and $\dot{x} > 0$. Because $x = 0$, momentarily there is no acceleration and the mass moves up (x increases). That is, the graph starts at the origin with a nearly straight piece sloped up and to the right. As x gets larger, the force pushing the mass back comes into play. The mass starts to slow down. The graph becomes less steep. At some point, the velocity goes through zero; the particle has gone as far from rest as it is going to go. This is a maximum for $x(t)$, a spot where the graph is momentarily horizontal. As the restorign force continues to act, the velocity becomes negative and the graph slopes downward. At the point where x comes back to zero, it's velocity is as large (negative) as it will ever be. When x crosses into the the negative, the restoring force starts pushing in the positive direction and it starts slowing down again. This paragraph will be most useful if you draw the picture while reading it.

Continuing in this way we see that the graph of $x(t)$ goes up and down like a sin curve. This motivates the *guess*[3]

$$x(t) = A\sin(\omega t) \; . \tag{5}$$

At this point, A and ω are unknown. To see whether this attempted solution satisfies the equation (4), we compute

$$v = \dot{x} = \frac{dx}{dt} = \omega A\cos(\omega t)$$

[2]There are many other oscillators in nature, ranging from swinging pendula to guitar strings to electrical circuits to the beating heart.

[3]The graph of the solution for a nonlinear oscillator could look similar even though the solution is not of this form.

$$\ddot{x} = \frac{dv}{dt} = -\omega^2 A \sin(\omega t) \ ,$$

so, comparing to (5), we see that our guess satisfies the equation

$$\ddot{x} = -\omega^2 x \ . \tag{6}$$

Now comparing (6) with the original (4), we see that (5) is indeed a solution, provided that

$$\omega = \sqrt{\frac{k}{m}} \ . \tag{7}$$

The significance of A is clear; A is the maximum displacement of the mass, the *amplitude* of the oscillation. The *period* of the oscillation is how long it takes to reach a point where it repeats itself exactly. The function $\sin(\theta)$ starts repeating when θ increases by 2π. Here $\theta = \omega t$, so the period, T is how much t has to increase in order that θ should increase by 2π: $T = 2\pi/\omega$. The number ω is the *angular frequency.* It determines the rate at which θ is increasing. The frequency, in cycles per unit time, is $1/T = \omega/2\pi$. In this class, we refer to ω, which is the number of radians per unit time, as the frequency and forget $\omega/2\pi$.

The frequency formula (7) has some very specific ("nontrivial" in mathematicians' jargon) consequences. For example, suppose you connect a mass to the wall using two identical springs. That has the effect of doubling the restoring force, replacing k with $2k$. This multiplies the frequency by $\sqrt{2} \approx 1.41$. If you were at a party asking what would happen, someone might guess that the frequency would double, but not many people would guess it increase by 41%. In the same way, if we double the mass without changing the spring constant, the frequency goes down by a factor of $\sqrt{2}$. If we double the mass and the spring constant the frequency does not change. We can understand this by thinking of two identical masses side by side connected to the same wall by two identical springs. They would undergo exactly the same motion whether or not they were connected. If they were connected, we would say we had doubled the mass and the spring constant.

Physicists have found that the concept of *energy* is very helpful in modeling physical systems. This is because energy takes many forms but often is conserved overall. In our context, there are two forms of energy, kinetic, written KE, and potential, written PE. The total energy is the sum: $E = KE + PE$. Physicists use the term "work" to mean the amount of energy you have added to a system. If an oscillator starts at rest and you push it to get it going, the energy it winds up with is the work you have done on the oscillator. Conversely, if you make it have less energy, it has done work on you.

Kinetic energy is the energy of motion.[4] It is given by

$$KE = \frac{1}{2} m v^2 \ . \tag{8}$$

If there were more than one mass or if the mass could move in more than one direction, the total kinetic energy would be the sum of the separate kinetic

[4] The word "kinetic" is related to the "cinema", where you see moving pictures.

energies of all the masses and over each possible direction of motion. Note that the kinetic energy is proportional to the mass. Adding an identical mass moving at the same speed doubles the kinetic energy. It is also proportional to the *square* of the velocity. Doubling the speed multiplies the kinetic energy by a factor of four. If W is the amount of work you have to do on a discus to make it leave your hand going ten feet per second, then $4W$ is what it takes to make it leave your hand going 20 feet per second.

Potential energy depends on x, not $v = \dot{x}$. It represents energy "stored" in the system, which is the result of pushing an object against a force. For the spring system, this force is the restoring force (2). The force and potential energy are related through

$$f = -\frac{d}{dx} PE(x) \; . \tag{9}$$

That is, the if you have to push harder (f on the left side is larger), you add potential energy to the system at a greater rate (the slope on the right side is larger). For the harmonic oscillator, we have

$$PE = \frac{1}{2} k x^2 \; . \tag{10}$$

You should check that this is consistent with (9) if you use the harmonic oscillator force (2).

For the solution (5), we have $v = \dot{x} = \omega A \cos(\omega t)$, so

$$KE = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t) \; ,$$

and

$$PE = \frac{1}{2} k A^2 \sin^2(\omega t) \; .$$

From the frequency formula (7), we have $m\omega^2 = k$, so the total energy is

$$\begin{aligned} E &= PE + KE \\ &= \frac{1}{2} k A^2 \sin^2(\omega t) + \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t) \\ &= \frac{1}{2} k A^2 \sin^2(\omega t) + \frac{1}{2} k A^2 \cos^2(\omega t) \\ &= \frac{1}{2} k A^2 \left(\sin^2(\omega t) + \cos^2(\omega t) \right) \\ E &= \frac{1}{2} k A^2 \; . \end{aligned} \tag{11}$$

This shows that the total energy in the harmonic oscillator does not change in time, even though the potential and kinetic energies go up and down. The phycisist's picture is that energy is constantly changing form, from kinetic to potential and back, without being created or distroyed. We also learn that the energy in a harmonic oscillator is proportional to the spring constant and proportional to the square of the amplitude. This is easy to see. When the displacement is as large as it gets, the velocity is zero. At that moment, $x = \sin(\omega t) = 1$ and the velocity is zero. That is, all the energy is in the form of potential energy. The potential energy at that moment is $\frac{1}{2}kx^2 = \frac{1}{2}k(A\sin(\omega t))^2 \frac{1}{2}kA^2$.

For future use, here is another way to verify that the total energy in the harmonic oscillator does not change with time. We compute the rate of change of total energy:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}PE + \frac{d}{dt}KE \\
&= \frac{d}{dt}\frac{1}{2}kx^2 + \frac{d}{dt}\frac{1}{2}m\dot{x}^2 \\
&= kx\dot{x} + m\dot{x}\ddot{x} \\
&= kx\dot{x} + m\dot{x}\left(-\frac{k}{m}x\right) \qquad \text{(using (4))} \\
&= kx\dot{x} - k\dot{x}x \\
&= 0
\end{aligned}$$

This derivation is useful because it allows us to calculate how the energy in a harmonic oscillator changes under the influence of outside forces other than the spring restoring force. If we call this "external" force f_e. A linear harmonic oscillator with an external force has total force $f = -kx + f_e$, so it's differential equation is

$$m\ddot{x} = -kx + f_e \ .$$

Repeating the calculation above gives

$$\frac{dE}{dt} = f_e \cdot \dot{x} \ . \tag{12}$$

I said above that the change in total energy of a system is the work done on it. That is, the work done on the oscillator between times t_1 and t_2 is $W = E(t_2) - E(t_1)$. A formula for this work comes from integrating (12):

$$W = \int_{t_1}^{t_2} f_e(t)\dot{x}(t)dt \ . \tag{13}$$

This formula has a simple interpretation. Suppose $t_2 = t_1 + dt$, where we think of dt as a small interval of time. The mass moves a distance $dx = \dot{x}dt$. The work done by pushing is the product of the force and the distance. Pushing twice as hard over the same distance doubles the work. No matter how hard you push, if the thing does not move, you have not done any work. The integral (13) add up $f \times$ *distance* over many small distances dx to get the total work over the time interval.

The effect of friction

As an exercise in physics, modeling, and approximation, we want to assess the effect of friction on a simple harmonic oscillator. Friction is a complicated physical phenomenon which, in the year 2000, is still not understood on the molecular level. There are several "models of friction" that are used in applied modeling of physical systems. The simplest model, from a mathematical point of view, is "simple linear friction". This is often at best a qualitative model of real friction, but it's a good place to start.

In simple linear friction, we assume that the friction force is proportional to the velocity of the mass, and pushing the opposite direction. That is

$$f_{fr} = -\gamma \dot{x} \tag{14}$$

To get the differential equation, we start with the original $ma = f$ equation (3) and put in the total force being the sum of the linear restoring force, $-kx$, and the friction force, $-\gamma\dot{x}$. This given the "linear damped oscillator" equation

$$m\ddot{x} = -kx - \gamma\dot{x} \ ,$$

or, after dividing through by m as before:[5]

$$\ddot{x} = -\frac{k}{m}x - \frac{\gamma}{m}\dot{x} \ . \tag{15}$$

With the physical model transformed into a mathematical model (15), we turn to mathematical approximation of the solution of the mathematical model. There are several ways to do this, but I want to start with a simple intuitive approach that has the disadvantage of seeming unsystematic but the advantage of being general (applying to other models of friction) and being clear physically. We assume that the damping coefficient is so small that is we just look at the oscillator in action, we might not notice at first glance that it is damped. We ignore the other extreme, the damping being so strong as to prevent oscillations. Mathematically, for not too long periods of time, the oscillator looks like $A\sin(\omega t)$. However, if we watch over longer times, we will notice the amplitude, A, decrease. We model this with the *ansatz*[6]

$$x(t) \approx A(t)\sin(\omega t) \ , \tag{16}$$

with the understanding that $\dot{A}$ is pretty small.

We are now going to use the ansatz (16) to estimate how much energy is lost in one full cycle of the oscillation. We will assume that A is constant during one full period of the oscillation, use that to estimate the work done by friction during that period, then figure out how much the amplitude decayed during that oscillation. To compute the work of friction during a period, we use the

[5] I think the equation I used in class was wrong, having just γ instead of γ/m as the coefficient of $\dot{x}$.

[6] Ansatz is a German word that does not seem to have any meaning outside of mathematics. In mathematics, it refers to an assumed form of the solution to an equation, possibly with unknown functions or parameters to be determined. I think of the word as being a combination of "hypothesis" and "starting point".

integral (13) with the friction force taking the role of external force. For simple linear friction (14), we have the work integral

$$\begin{aligned} W &= \int f_{fr}(t)\dot{x}(t)dt \\ &= -\gamma \int \dot{x}^2(t)dt \; . \end{aligned}$$

Now for the approximation. Using $x(t) = A\sin(\omega t)$ we get $\dot{x} \approx \omega A \cos(\omega t)$. As I said above, this is because we neglect the other term $\dot{A}\sin(\omega t)$ which is supposed to be much smaller than the one we're keeping. With this, the work integral becomes

$$W \approx -\gamma\omega^2 A^2 \int \cos^2(\omega t)dt \; .$$

Let us call the period T. We saw that $T = 2\pi/\omega$, but that is not too important here. The important thing is that the average value of $\cos^2$ over a full period of cos is[7] $1/2$. This gives the energy loss over a period as

$$\Delta E = -\gamma\omega^2 A^2 \cdot T \cdot \frac{1}{2} \; .$$

From the energy formula (11) we see that

$$\Delta E = \frac{1}{2} k \Delta \left(A^2\right) \approx kA\Delta A \; .$$

Finally, equating these two expressions involving A, we get

$$kA\Delta A = -\gamma\omega^2 A^2 \cdot T \cdot \frac{1}{2} \; ,$$

which, after some algebra, is the same as

$$\frac{\Delta A}{T} = -\frac{1}{2}\frac{\gamma\omega^2}{k} A \; .$$

Now, we were supposing that ΔA is small, relative to A, over the time period T. For that reason, we may write

$$\frac{\Delta A}{T} \approx \frac{dA}{dt} \; .$$

Also, the frequency relation (7) allows us to simplify the left side to γm. Altogether, we get

$$\frac{dA}{dt} = -\frac{\gamma m}{2} A \; . \tag{17}$$

[7] You can see this from the graph, or using the trigonometric identity $\cos^2(\theta) = 1/2 + 1/2\cos(2\theta)$. Actually, the trig identity is also obvious from the graph.

The solution to the differential equation$\dot{A} = -\mu A$ is $A(t) = A(0)e^{-\mu t}$ (check this), so the simple linear friction model, at least when the friction is weak, predicts that the amplitude decays as a simple exponential:

$$A(t) = A(0)e^{-\gamma mt/2} \ . \tag{18}$$

In the next class, we will see how accurate this rough and ready approximation is.