## Binomial Coefficients

Pascal's Triangle. The approach we take is different from the text, though we end in the same place. Pascal's Triangle is formed by starting with a single number 1, and building the "triangle" with that as a starting point:

To build the triangle, think of this 1 as preceded by 0's and followed by zeros. We will never write them in, but this is what we have in mind.
... 0001000 ...
We now build the second row of numbers using the following rule:

For any two consecutive number in a row, place their sum under the second of these numbers.

In this way we obtain two rows (in which we have not written in the zeros.)

$$
\begin{array}{ll}
1 & \\
1 & 1
\end{array}
$$

Now do it again to form a third row, using the same rule:

| 1 |  |  |
| :--- | :--- | :--- |
| 1 | 1 |  |
| 1 | 2 | 1 |

And again:

```
1
1 1
1 2 1
1
```

In the more classical Pascal Triangle this is written as

in which the formation rule is changed to place the sum of two consecutive numbers on a row by their sum underneath and between them. However, we shall stick to the original law of formation. If we continue this process, we arrive at a table of arbitrary length. Stopping at 7 lines, we get

| 1 |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 |  |  |  |  |  |
| 1 | 2 | 1 |  |  |  |  |
| 1 | 3 | 3 | 1 |  |  |  |
| 1 | 4 | 6 | 4 | 1 |  |  |
| 1 | 5 | 10 | 10 | 5 | 1 |  |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 |

This table will be immediately recognized by every mathematician. The numbers in this table are called binomial coefficients for reasons we shall see later. We have underlined one of the numbers 10. It is in the 5th row and 2nd column. (Careful here, since the top row is called the 0 -th row, and the left-most column the 0 -th column!) We write

$$
10=\binom{5}{2}
$$

We can label the columns and rows to see this more clearly:

| Row | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  |  |
| 1 | 1 | 1 |  |  |  |  |  |
| 2 | 1 | 2 | 1 |  |  |  |  |
| 3 | 1 | 3 | 3 | 1 |  |  |  |
| 4 | 1 | 4 | 6 | 4 | 1 |  |  |
| 5 | 1 | 5 | $\underline{10}$ | 10 | 5 | 1 |  |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

The symbol $\binom{n}{r}$ is called ' $n$ over $r$ ', or ' $n$ choose $r$.' From the above table, you should be able to see that $\binom{4}{2}=6$ and $\binom{5}{6}=0$.

We can review the construction of this table more algebraically using the notation $\binom{n}{r}$. For the start, we had a 1 on the top row, with 0's (understood) to the right and the left of it. Thus, we started the process using

$$
\binom{0}{0}=1 ;\binom{0}{r}=0 \text { for } r \neq 0
$$

The rule for placing a number on a new row was "For any two consecutive number in a row, place their sum under the second of these numbers." Thus, if we are in row $n$ and columns $r$ and $r+1$, with entries, $\binom{n}{r}$ and $\binom{n+1}{r}$ respectively, we go to row $r+1$ and define

$$
\binom{n+1}{r+1}=\binom{n}{r}+\binom{n+1}{r}
$$

Thus, summarizing, the numbers $\binom{n}{r}$ are defined for all $n \geq 0$ and all $r$ by the two conditions:

$$
\begin{equation*}
\binom{0}{0}=1 ;\binom{0}{r}=0 \text { for } r \neq 0: \text { (The initial conditions) } \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\binom{n+1}{r+1}=\binom{n}{r}+\binom{n+1}{r} \text { for all } r \text { and all } n \geq 0: \text { (The recursion) } \tag{2}
\end{equation*}
$$

Equations (1) are called the initial conditions, since they start (initialize) the construction process. A recursion is any method used to compute new values from already computed values. Thus, Equation (2) is properly called a recursion. Equations (1) and (2) do not give a formula for $\binom{n}{r}$. Rather, they allow you to systematically find it by computing row by row. To find $\binom{20}{10}$ you would have to compute 20 additional rows in the manner described above and find out what number is in the 20th row and 10th column. This is systematic, but long. (In fact, $\binom{20}{10}=184,756$.) We shall find a formula for $\binom{n}{r}$ in what follows. We can see from the table, however, that $\binom{n}{0}=1$ and $\binom{n}{1}=n$.

A critical fact that we shall use is that the initial conditions (1) and the recursion (2) completely determine the values of $\binom{n}{r}$.

The Binomial Theorem. In algebra, a binomial is simply a sum of two terms. For simplicity, we shall work with the binomial $1+x$. The Binomial Theorem is a formula which
computes a power of a binomial. Most of us are familiar with the formula $(1+x)^{2}=1+2 x+x^{2}$. The coefficients here are 1, 2, 1-precisely the numbers in the second row of Pascal's triangle. Note also that $(1+x)^{1}=1+x$ and the coefficients are the first row of Pascal's triangle. Why should this be? Let's check the situation for $(1+x)^{3}$. We assume you don't know the formula, though you might be able to guess it at this point. We have

$$
(1+x)^{3}=(1+x)^{2}(1+x)=(1+x)^{2}+x(1+x)^{2} .
$$

Let's compute in tabular form

$$
\begin{aligned}
&(1+x)^{2}=1+2 x+x^{2} \\
& x(1+x)^{2}= \\
& x+2 x^{2}+x^{3} \\
& \hline \text { Adding, }(1+x)^{3}=1+3 x+3 x^{2}+x^{3}
\end{aligned}
$$

Now that we have the formula for $(1+x)^{3}$, let's use the same idea with

$$
\begin{aligned}
&(1+x)^{4}=(1+x)^{3}(1+x)=(1+x)^{3}+x(1+x)^{3} . \\
&(1+x)^{3}=1+3 x+3 x^{2}+x^{3} \\
& x(1+x)^{3}= \\
& x+3 x^{2}+3 x^{3}+x^{4} \\
& \hline \text { Adding, }(1+x)^{4}=1+4 x+4 x^{2}+4 x^{3}+x^{4}
\end{aligned}
$$

Thus, for $(1+x)^{4}$, the coefficient of any power of $x$ is seen to the sum of the coefficients of that power and the preceding power in the expansion of $(1+x)^{3}$. In other words, the coefficients are obtained by recursion generating Pascal's triangle. This process can continue to find the coefficients of $(1+x)^{5}$ as the fifth row of Pascal's triangle, and so on. For a starting point (the initial condition) we can choose $(1+x)^{0}=1$, the same starting point as in Pascal's triangle. This method gives us the Binomial Theorem:

$$
(1+x)^{n}=\binom{n}{0}+\binom{n}{1} x+\binom{n}{2} x^{2}+\ldots+\binom{n}{n} x^{n}=\sum_{r=0}^{n}\binom{n}{r} x^{r}
$$

A similar proof gives another version of the Binomial Theorem for a more general binomial ${ }^{1}$ :

$$
(x+y)^{n}=\binom{n}{0} x^{n}+\binom{n}{1} x^{n} y+\binom{n}{2} x^{n-2} y^{2}+\ldots+\binom{n}{n} y^{n}=\sum_{r=0}^{n}\binom{n}{r} x^{n-r} y^{r}
$$

In this second formulation, make sure that the power of $x$ and of $y$ add up to $n$. For example, for $n=5$,

$$
(x+y)^{5}=x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+y^{5}
$$

[^0]Counting Choices. Suppose we have five objects and we want to choose two of them. How many ways can we do this? If the objects are numbered $\{1,2,3,4,5\}$, we can actually list the possibilities. They are ${ }^{2}$ :

12131415232425343545

Therefore there are ten ways to choose two objects from five.
Note that $\binom{5}{2}=10$. It turns out that in the general problem where we choose $r$ objects from among $n$, there are $\binom{n}{r}$ ways of doing this. Why is this so?

Suppose we let $C(n, r)$ be the number of ways of choosing $r$ objects from $n$. We have just computed that $C(5,2)=\binom{5}{2}=10$. We now want to show that $C(n, r)=\binom{n}{r}$. There is a nice tricky way of doing this. In the above list of 10 choices, let us separate the ones where 1 is chosen from the ones where 1 is not chosen. Thus, we have two groups:

In the first group, where 1 is already chosen, we have a choice of one of the remaining four other numbers, so we have $4=C(4,1)$ choices in all. In the second group, where 1 is not to be chosen, we have to choose two from among the remaining four, with $C(4,2)=6$ choices in all. The important consideration here is that

$$
C(5,2)=C(4,1)+C(4,2)
$$

In general, this argument show that

$$
C(n+1, r+1)=C(n, r)+C(n, r+1)
$$

for if the objects are the numbers from 1 through $n+1$, we can choose $r+1$ from them by either including 1 or not. And the number of choices respectively are $C(n, r)$ and $C(n, r+1)$ respectively. This argument shows that the numbers $C(n, r)$ satisfy the same recursion relation as the numbers $\binom{n}{r}$. The initial condition $C(0,0)=0$ is a bit obscure ${ }^{3}$ : If there are no objects $(n=0)$ there is exactly one way of choosing no objects $(r=0)$ from them, so

[^1]$C(0,0)=1$, but there are no ways of choosing 1 or more objects from them, or a negative number of objects from them. $(C(0, r)=0$ if $r \neq 0$.) Therefore, since the numbers $C(n, r)$ and $\binom{n}{r}$ have the same initial conditions and satisfy the same recursion relation, they are equal:
$$
C(n, r)=\binom{n}{r}
$$

The Factorial Formula. The symbol $n$ ! is used to represent the product $1 \cdot 2 \cdot 3 \cdots n$. $n$ ! is read $n$ factorial. For example, $4!=1 \cdot 2 \cdot 3 \cdot 4=24$. It is also convenient to define $0!=1$. This is a function which can be defined recursively by giving it an initial value and defining its value in terms of a

$$
\begin{equation*}
0!=1 \text { (Initial value) } \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
(n+1)!=(n+1) n!\text { for all } n \geq 0 \text { (Recursion) } \tag{4}
\end{equation*}
$$

To see how this works, put $n=0$ in (4) to get $1!=1 \cdot 0!=1$ by (3). Now put $n=1$ in (4) to get $2!=2 \cdot 1!=2$ since we found $1!=1$. Continuing by putting $n=2$ in (4) to get $3!=3 \cdot 2!=3 \cdot 2=6$ since we found $2!=2$. We can clearly continue to get $4!=4 \cdot 6=24$, $5!=5 \cdot 24=120$ and so on. Equations (3) and (4) constitute the recursive definition of $n!$, and it is consistent with the product definition given above. Warning: these numbers grow very fast. For example, $20!=2.43 \times 10^{18}$.

Using the factorial function, we can find a useful formula for $\binom{n}{r}$. Let's look more carefully at Pascal's triangle.

| Row | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  |  |
| 1 | 1 | 1 |  |  |  |  |  |
| 2 | 1 | 2 | 1 |  |  |  |  |
| 3 | 1 | 3 | 3 | 1 |  |  |  |
| 4 | 1 | 4 | 6 | 4 | 1 |  |  |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |  |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

The first column consists of 1's only: Thus $\binom{n}{0}=1$. The second column is equally clear: $\binom{n}{1}=n$. We now investigate the second column. The numbers are as follows:

$$
\begin{array}{r|rrrrrrr}
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline C(n, 2) & 0 & 0 & 1 & 3 & 6 & 10 & 15
\end{array}
$$

A formula which works for this table is $C(n, 2)=n(n-1) / 2$. This is not a guess out of left field. We note that $6=1+2+3$, and $10=1+2+3+4$, and we can use the formula for summing an arithmetic series to derive this ${ }^{4}$. At the moment, however, this is a guess and we have yet to check it even for $n=7 .{ }^{5}$

It turns out that a similar formula works for the third column. It is

$$
\binom{n}{3}=\frac{n(n-1)(n-2)}{6}
$$

For example, $\binom{5}{3}=\frac{5 \cdot 4 \cdot 3}{6}=10$. The above formula harbors factorials. Thus, (since $6=3$ !),

$$
\binom{n}{3}=\frac{n(n-1)(n-2)}{6}=\frac{n(n-1)(n-2)}{3!} \cdot \frac{(n-3)!}{(n-3)!}=\frac{n!}{3!(n-3)!}
$$

Going one step further, we can conjecture:

$$
\begin{equation*}
\binom{n}{r}=\frac{n!}{r!(n-r)!} \tag{5}
\end{equation*}
$$

Formula (5) is in fact true, but is at this point we have given no proof. How can we now prove it? Let us show that the expression

$$
F(n, r)=\frac{n!}{r!(n-r)!}
$$

satisfies the same initial conditions and recursion formula as in Pascal's triangle. For starters, $F(0,0)=\frac{0!}{0!0!}=1 .{ }^{6}$ The recursion itself requires some algebra and is essentially covered in the Koshy text, page 35 . We repeat it on the following page in our notation. The calculation needs little more than a knowledge of how to add fractions and the properties of factorials.

[^2]\[

$$
\begin{aligned}
F(n, r)+F(n, r+1) & =\frac{n!}{r!(n-r)!}+\frac{n!}{(r+1)!(n-r-1)!} \\
& =\frac{(r+1) n!}{(r+1)!(n-r)!}+\frac{n!(n-r)}{(r+1)!(n-r)!} \\
& =\frac{(r+1) n!+n!(n-r)}{(r+1)!(n-r)!} \\
& =\frac{n!(r+1+n-r)}{(r+1)!(n-r)!} \\
& =\frac{n!(n+1)}{(r+1)!(n-r)!} \\
& =\frac{(n+1)!}{(r+1)!(n-r)!} \\
& =F(n+1, r+1)
\end{aligned}
$$
\]

This section tied together four concepts: Pascal's triangle, the Binomial Theorem, Counting the number of choices taking $r$ objects from $n$, and a formula for these numbers.

Some Properties of $\binom{\mathbf{n}}{\mathbf{r}}$. For typographical reasons, we shall use $C(n, r)$ instead of $\binom{n}{r}$. Formula (5) has a simple consequence. We can illustrate it numerically, by noting that by this formula,

$$
C(12,9)=\frac{12!}{9!3!}, \text { and } C(12,3)=\frac{12!}{3!9!} \text { so } C(12,9)=C(12,3)
$$

In general, we have

$$
\begin{equation*}
C(n, r)=C(n, n-r) \tag{6}
\end{equation*}
$$

The general proof uses Formula (5). Since $C(n, r)=\frac{n!}{n!(n-r)!}$, we have

$$
C(n, n-r)=\frac{n!}{(n-r)!(n-(n-r))!}=\frac{n!}{(n-r)!r!}=C(n, r)
$$

This is the mathematical way of noting that any row of Pascal's triangle reads the same backwards as well as forward.

If we add the rows of Pascal's triangle we arrive at the following table

| Row | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Sum |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  | 1 |  |
| 1 | 1 | 1 |  |  |  |  | 2 |  |
| 2 | 1 | 2 | 1 |  |  |  | 4 |  |
| 3 | 1 | 3 | 3 | 1 |  |  | 8 |  |
| 4 | 1 | 4 | 6 | 4 | 1 |  |  | 16 |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |  | 32 |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | 64 |

You are correct in thinking this is no accident. In general the numbers in row $n$ add up to $2^{n}$. We can prove this using the Binomial Theorem:

$$
(1+x)^{n}=C(n, 0)+C(n, 1) x+C(n, 2) x^{2}+\cdots+C(n, n) x^{n}
$$

Substitute $x=1$ to get the result:

$$
2^{n}=C(n, 0)+C(n, 1)+C(n, 2)+\cdots+C(n, n)
$$

We end by noting that Formula (5), while useful for theoretical purposes, is not very practical for calculations. For example, to compute $C(12,9)$ using this formula, we would find

$$
C(12,9)=\frac{12!}{9!3!}=\frac{479001600}{362880 \cdot 6}=220
$$

A simpler way is to note that

$$
\frac{12!}{9!3!}=\frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1}=220
$$

by observation. In general to do a hand calculation of $C(n, r)$, we can compute

$$
C(n, r)=\frac{n(n-1) \cdots(n-r+1)}{r!}
$$

(Note that the numerator has exactly $r$ factors.) This can be shortened if $n-r<r$ by using $C(n, r)=C(n, n-r)$. For example, to find $C(101,99)$, we compute

$$
C(101,99)=C(101,9)=101 \cdot 100 / 2=5050
$$


[^0]:    ${ }^{1}$ Another proof is not necessary. You can replace $x$ by $y / x$ and multiply through by $x^{n}$.

[^1]:    ${ }^{2}$ Note that we regard 12 and 21 as the same choice. We are concerned with the choices themselves, and not the order in which these choices are chosen.
    ${ }^{3}$ Obscure, because we are dealing in a serious way with the empty set containing 0 objects.

[^2]:    ${ }^{4}$ The fact that $C(n, 2)=0$ for $n=0$ and $n=1$ suggests the possibility of using $n(n-1)$ in the formula.
    ${ }^{5}$ It works: $\binom{7}{2}=21=\frac{7 \cdot 6}{2}$.
    ${ }^{6}$ The initial row of mostly 0 's is taken care of by the convention that the formula for $F$ gives 0 whenever we have the factorial of a negative number.

