Cauchy Problem

We start with:

1. Take \( x_0 \) and \( t_0 \)

2. For each \( x \neq x_0 \) and \( t \neq t_0 \) consider \( n = \min \{ x, t \} \)

3. Define \( q(t) = \delta \)

4. Define \( \Phi(t) = \Phi(t_{k-1}) + \int_{t_{k-1}}^{t} \Phi(t_{k-1}) \Phi(t_{k}) \) \((t < t_{k} = k \in \mathbb{N})\)

5. Then \( \Phi(t) = \Phi(t_{k-1}) + \int_{t_{k-1}}^{t} \Phi(t_{k-1}) \Phi(t_{k}) \) \((t < t_{k} = k \in \mathbb{N})\)

6. Hence \( \frac{1}{\Phi_{n}(t) - \Phi_{n-1}(t)} \leq e^{nt} \) so \( \Phi(t) \) is bounded by \( e^{nt} + b \)

7. Equicontinuous

8. A.A. \( \Rightarrow \) uniformly converges: 

9. \( \Phi(t) \) is a solution

10. Subsequence of \( \{ u_n \} \) is unique

11. \( \lim \Phi(t) = \Phi(t) \)

Proof: Uniformly approximated

1. Assume \( \Phi(t) = \Phi(t_{n}) \)

2. Integrals \( \gamma_{n}(t) = \int_{0}^{t} f(s, y_{n}(s)) ds \)

3. Assume \( \beta_{n}(t) \leq \int_{0}^{t} \beta_{n}(s) ds \)

4. \( \beta_{n}(t) = \beta_{n}(t_{n}) \leq \left( t_{n} - t \right)^{n-1} \beta_{n}(t_{n}) \)

5. \( \gamma_{n}(t) = \beta_{n}(t) \leq \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{0}^{t} \beta_{n}(s) ds \)

6. Hence \( \gamma_{n}(t) \) is uniformly bounded.

7. Therefore, \( \{ u_{n} \} \) converges uniformly.

8. \( \exists \lim \Phi(t) = \Phi(t) \)

9. \( \Phi(t) \) is unique.

10. \( e^{nt} + b \) as \( n \to \infty \)
Existence & Uniqueness

\[ \psi(C) \quad \text{s.t.} \quad \circ \,(t,\psi(t)) \quad t < 1 \]
\[ \Omega \psi(C) = \psi \quad \text{with} \quad \psi = \min(a, b) \]

Theorem: If \( C \in R \) then \( \psi(C) \) exists.

Proof: By completeness of \( \psi(C) \) and \( \psi = \min(a, b) \), we have that \( \psi(C) \) exists.

\[ \psi(C) = \psi \quad \text{s.t.} \quad \psi(t) = \min(a, b) \]

Candy Plane: If \( C \in R \), then \( \psi(C) \) exists with \( \psi(C) \).

Proof: By completeness of \( \psi \).

Choose a subsequence \( \psi \) that converges by \( k \).

Example: Choose a subsequence \( \psi \) that converges.

Uniqueness: Choose a subsequence and ensuring.

Example: Choose a subsequence ensuring even though uniqueness not satisfied.
ODE
Equation & Uniqueness

\[ \frac{dx}{dt} = f(x) \]  
\[ \frac{dy}{dt} = g(y) \]

Prove: Two stages:
1. By contradiction, show approximate solutions can't overlap.
2. Derive a sequence of approximate solutions that tend to solutions.

**Existence and Uniqueness Theorem**

**Proof 1:**

**Proof 2:**

**Existence:**

- **Sufficient Condition:**
  - \( f(x) \) is continuous and \( g(y) \) is continuous.
  - \( f(x) \) and \( g(y) \) are Lipschitz continuous.

**Uniqueness:**

- **Lipschitz Condition:**
  - \( f(x) \) and \( g(y) \) satisfy the Lipschitz condition.

**Proof:**

**Sufficient Condition:**

- **Existence:**
  - By the Picard-Lindelöf theorem.

**Uniqueness:**

- **Proof:**
  - By contradiction.
  - Assume two different solutions.

**Picard-Lindelöf Theorem:**

- **Existence:**
  - Continuous in initial conditions.

**Successive Approximations:**

- **Convergence:**
  - Monotonic.

**Existence Based on Induction:**

- **Finite differences:**
  - \( \Delta x = \frac{x_{n+1} - x_n}{h} \)
  - \( \Delta y = \frac{y_{n+1} - y_n}{h} \)

**Proof by Induction:**

- **Base Case:**
  - \( n = 0 \)

- **Inductive Step:**
  - Assume \( x_0, y_0 \) is unique.

**Successive Approximations:**

- **Convergence:**
  - To a unique solution.
Stability of an Equilibrium Point (linear analysis, i.e., a linear solution)

① It is stable if the eigenvalues are all negative.
② It is asymptotically stable if it is not only stable but also the solution converges to the equilibrium point.

Definition: A point in the state space is an equilibrium point if the system's state remains unchanged as time progresses. Mathematically, if \( \dot{x} = 0 \) for all \( t \), then \( x \) is an equilibrium point.

Nonlinear Eq. Solved

The equation is solved for \( x \) when the system is linearized around an equilibrium point. The linearized equation is obtained by approximating the nonlinear terms with their Taylor series expansions around the equilibrium point.

\[ \dot{x} = A x + B u \]

where \( A \) is the Jacobian matrix of the nonlinear function \( f(x) \) at the equilibrium point, \( B \) is the input matrix, and \( u \) is the input vector.

Eigenvalues of the Jacobian matrix \( A \) determine the stability of the equilibrium point. If all eigenvalues have negative real parts, the equilibrium point is asymptotically stable.

Example: Consider the linearized system

\[ \dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

The eigenvalues of the Jacobian matrix are \( -1 \pm i \), indicating that the equilibrium point is a center.

Further analysis of the stability may require additional conditions and techniques, such as Lyapunov functions or center manifold theory.
**Examples**

\[ \dot{x} = -x \quad f(x) = ke^{-x} = \dot{x} \]

**Plane Points**

- Family of all solutions are subsets of \( \mathbb{R}^2 \) is called the "plane points".

**Dynamical Systems**

- Solution as \( \phi(t) \)
  \[ \phi(t) = \Phi(x,0,t) \]
- For fixed \( x \), transformation \( \Phi(x) \)
  \[ \Phi(x) : \mathbb{R}^2 \to \mathbb{R}^2 \]
- Collection of maps in \( \Phi \) parameter family of transformations. Call the flow or dynamical systems.

\[ \Phi(t) = \Phi(\alpha t) \quad \alpha \in \mathbb{R} \]

\[ \phi(x) = \Phi(x,0,t) \]

\[ \Phi(x) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \]

**Chapter 2**

**Harmonic Oscillator**

- Force \( -mp^2 = -x_0 \cos(x) \)
  \[ \ddot{x} + \omega_0^2 x = 0 \]

- Simple Pendulum
  \[ \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0 \]

- Simple Spring
  \[ \frac{d^2x}{dt^2} + kx = 0 \]

**Conservative Force Field**

- Force \( F(x) = -V'(x) \)

- Equipotential lines \( \nabla V = 0 \)

**General Force Field**

- Force \( F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} \)
  \[ F_1(x) = f_1(x) \]
  \[ F_2(x) = -V'(x) \]
  \[ V(x) = g(x) \]
Continuing on EC.

Use Gronwall's inequality, \( u(t) \leq C + \int_{0}^{t} e^{(t-s)} \, ds \), \( t \in \mathbb{R}^+ \), \( C \geq 0, t \geq 0 \).

\[ u(t) \leq C + \int_{0}^{t} e^{(t-s)} \, ds \leq C + \int_{0}^{t} e^{t} \, ds = C + e^{t} \int_{0}^{1} e^{-s} \, ds = C + e^{t} (1 - e^{-t}) \]

\[ u(t) \leq C + e^{t} (1 - e^{-t}) \]

\[ u(t) \leq C + e^{t} (1 - e^{-t}) \leq C + e^{t} \] }
Linear Homogeneous \( x' = A(t)x \)

- Set of solution \( f \) n-dimensional vector space
- Initial condition of fundamental matrix \( \Phi(t) \)
- \( \Phi(t) \in \mathbb{C} \)

- Adjoint Systems \( x' = A^*(t)x \) \[ \Phi(t) = e^{\int A^*(t)dt} \]
  \[ x' = A(t)x \quad \Rightarrow \quad x = \Phi(t)z \]
  \[ x' = A^*(t)x \]

- Reduction of order
\[ x'(t) = A(t) x(t) \]

\[ \textbf{b}(t) \]

\[ (dx + \textbf{b})' = x + \textbf{b}(t) \quad \text{or} \quad \frac{dx}{dt} = x + \textbf{b}(t) \]

\[ \int_2^t \frac{dx}{x} \text{ d}t \]
To maintain planar dynamical systems

If \( \frac{d}{dt} x = f(x, \mathbf{y}) \), then \( \mathbf{y}(t + h) = \mathbf{y}(t) + h f(x(t), \mathbf{y}(t)) \)

**Local Existence**

Let \( \mathbb{E} \) be an open set \( S \) containing \( 0 \) such that \( \mathbb{E} \) is a domain of \( \Phi \).

Flow \( \Phi^t \) is the exponential map \( \Phi^t : \mathbb{E} \rightarrow \mathbb{E} \) defined by \( \Phi^t(x) = \Phi(x, t) \).

**Stability**

Flow \( \Phi^t \) is stable if for every \( x \in \mathbb{E} \), the orbit \( \Phi^t(x) \) converges to an equilibrium point \( x_0 \).

**Local Bifurcation**

If \( \Phi(x, 0) = x \) and \( \frac{d}{dt} \Phi(x, t) = f(x, \mathbf{y}(t)) \)

\( \mathbf{y}(t) = \mathbf{y}_0 + h f(x(t), \mathbf{y}(t)) \)
\[ x = A(t)x \quad A(t) = A(t + T) \quad \mathbb{E}(t) FM = 0 \quad \mathbb{E}(t + T) FM \]

\[ \mathbb{E}(t + T) = \mathbb{E}C = \mathbb{E}(t) e^{T/2} \]

\[ P(t) = \mathbb{E}e^{-T/2} \]

\[ P(t + T) = \mathbb{E}(t + T) e^{-T/2} = \mathbb{E}(t) e^{-T/2} e^{T} \]

\[ = \mathbb{E}(t) e^{T} = P(t) \]

\[ \mathbb{E}(t) - P(t)e^{T/2} \quad \text{Proof:} \]

Characteristic \( e^{(\lambda)R} \) multipliers.

Eigenvalues \( \lambda \in \mathbb{R} \)

Phase Exponents \& Eigenvalues of \( \lambda \quad \text{unique and } \lambda \neq 0 \)
EXISTENCE OF SOLUTIONS TO ODE's

Math118, O. Knill

ABSTRACT. This is a proof of local existence of solutions of ordinary differential equations.

METRIC SPACES. Let $X$ be a set on which a distance $d(x, y)$ between any two points $x, y$ is defined. The function $d$ must have the properties $d(x, y) = d(y, x) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and that $d(x, y) > 0$ for two different points $x, y$. Furthermore, one requires the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ to hold for all $x, y, z$. A pair $(X, d)$ with these properties is called a metric space.

EXAMPLES. 1) The plane $\mathbb{R}^2$ with the usual distance $d(x, y) = |x - y|$. An other metric is the Manhattan or taxicab metric $d(x, y) = (x_1 - y_1) + (|x_2 - y_2|)$.

2) The set $C[0, 1]$ of all continuous functions $x(t)$ on the interval $[0, 1]$ with the distance $d(x, y)$ = max$|x(t) - y(t)|$ is a metric space.

CONTRACTION. A map $f : X \rightarrow X$ is called a contraction, if there exists $k < 1$ such that $d(f(x), f(y)) \leq k d(x, y)$ for all $x, y \in X$. The maps $f$ shrink the distance of any two points by the contraction factor $k$. EXAMPLES. 1) The map $f(x) = 1/2 + 1/(1 + 0)$ is a contraction on $\mathbb{R}$.

2) The map $f(x) = \sin(x(t)) + t$ is a contraction on $C[0, 1]$ because $|\sin(x(t)) - \sin(y(t))| \leq |x(t) - y(t)|$ and $|x(t) - y(t)| \leq |x(t) - f(t)|$.

CAUCHY SEQUENCE. A Cauchy sequence in a metric space $(X, d)$ is defined to be a sequence which has the property that for any $\varepsilon > 0$, there exists $n_0$ such that $d(x_n, x_m) < \varepsilon$ for $n, m > n_0$. EXAMPLES. 1) $(0^\infty, d(x, y) = |x - y|)$ is complete. The rational numbers $Q$, with $d(x, y) = |x - y|$ are not.

2) $(C[0, 1], d)$ is complete: given a Cauchy sequence $x_n(t)$ on $X$, then $x(t) = \lim x_n(t)$ is Cauchy sequence in $R$ for all $t$. Therefore $x_n(t)$ converges point-wise to a function $x(t)$. This function $x(t)$ is continuous, taking $r > 0$, then $|x(t) - x(t)| < |x_n(t) - x(t)| + |x_n(t) - x(t)| + |x(t) - x(t)|$ by the triangle inequality. For all $t < 0$, the second term is smaller than $\varepsilon$. For large $n$, $|x_n(t) - x(t)| < 2\varepsilon$ and $|x_n(t) - x(t)| < |x_n(t) - x(t)| < |x_n(t) - x(t)| < 2\varepsilon$. So, $|x(t) - x(t)| < \varepsilon$ if $\varepsilon$ is small.

COMPLETENESS. A metric space in which every Cauchy sequence converges to a limit is called complete.

BANACH’S FIXED POINT THEOREM. A contraction $f$ in a complete metric space $(X, d)$ has exactly one fixed point in $X$.

PROOF. (i) We first show by induction that $d(n^\infty(x), n^\infty(y)) \leq \lambda^n d(x, y)$ for all $n$.

(ii) Using the triangle inequality and $\lambda = 1 - \lambda$, we get for all $n \in X$.

$$d(n^\infty(x), n^\infty(y)) \leq \lambda^n d(x, y)$$

(iii) For all $x$ in the sequence $x_n = f(x)$ because by (i), (ii).

(iv) By completeness of $X$ it has a limit $x$ which is a fixed point of $f$.

(v) There is only one fixed point. Assume, there were two fixed points $x$, $y$. Then

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y).$$

This is impossible unless $x = y$.

THE CAUCHY-PICARD EXISTENCE THEOREM. Assume $f : R^2 \rightarrow R^1$ has a continuous derivative. For every initial condition $x_0$ there exists $r > 0$ such that on the time interval $[0, r)$ there exists exactly one solution of the initial value problem

$$x(t) = f(x(t)), x(0) = x_0$$

PROOF. (i) Consider for $r > 0$ and $r > 0$ the complete metric space $X = X(r) = \{x \in C[0, r] \mid |x(t)| \leq r \}$ with metric $d(x, y) = \max_{t \in [0, r]} |x(t) - y(t)|$. With $c(t) = x_0$ we can write also $X = \{x \mid d(x, c) \leq r\}$

Define a map $\phi$ on $[0, r]$ by

$$\phi(t) = t + \int_0^t d(x(s), c) ds$$

(ii) Define the constant

$$L = \max \left[ \frac{|f(x(t)) - f(c(t))|}{|x(t) - c(t)|}, \frac{|x(t) - c(t)|}{|x(t) - c(t)|} \right] \leq \lambda \cdot d(x, y)$$

for every $0 < r < r$.

Therefore

$$d(x(t), c(t)) = \max_{t \in [0, r]} \left| \int_0^t f(x(s)) ds - f(x(s)) ds \right| \leq \int_0^r \left| f(x(s)) ds - f(x(s)) ds \right| ds \leq L \cdot d(x, c)$$

We see that for small enough $r$, the map $\phi$ is a contraction.

(iii) With $N = \max \{|f(t)|/(|x(s)| - |c(s)|) \mid s \leq t, d(x, c) \leq 1, n \neq 1\}$, we have

$$|\phi(t) - c(t)| = \int_0^t f(x(s)) ds \leq \int_0^t |f(x(s)) ds| ds \leq M - \varepsilon$$

If $\varepsilon = 1$ is small enough, then $M \cdot \varepsilon < (1 - \lambda)\varepsilon$. Using the triangle inequality, we obtain

$$d(x(t), c(t)) = d(x(t), c(t)) + d(x(t), c(t)) \geq d(x(t), c(t)) + M \cdot \varepsilon < (1 - \lambda)\varepsilon$$

proving that $\phi$ maps $X = \{x(s, c) \leq r\}$ into itself.

(iv) The fixed point $x$ in $X$ obtained by Banach’s fixed point theorem is the solution of the differential equation $x(t) = f(x(t))$ with initial value $x(0) = x_0$.

EXAMPLE WITH NO UNIQUE SOLUTION. The differential equation $\dot{x} = \sqrt{x}$ with initial condition $x(0) = 0$ has the solution $x(t) = C^{1/2} t$ for any $C$. There are infinitely many solutions with the initial condition $x(0) = 0$.

EXAMPLE WITH NO GLOBAL SOLUTION. The differential equation $\dot{x} = x^2$ with initial condition $x(0) = 1$ has the solution $x(t) = 1/\left(1 - \beta t \right)$. At $t = 1$, the solution has escaped to infinity.

P.S. The photos show Stefan Banach (1892-1945), Edouard Picard (1856-1941) and Augustin Cournot (1796-1857).
**THE POINCARE BENDIXON THEOREM**

**Math18, O. Knill**

**ABSTRACT.** The Poincaré-Bendixon theorem tells that the fate of any bounded solution of a differential equation in the plane is to converge either to an attracting fixed point or to a limit cycle. This theorem rules out "chaos" for differential equations in the plane.

**THEOREM (Poincaré-Bendixon).** Given a differential equation \( \dot{x} = F(x) \) in the plane. Assume \( x(t) \) is a solution curve which stays in a bounded region. Then either \( x(t) \) converges for \( t \to \infty \) to an equilibrium point where \( F(x) = 0 \), or it converges to a simple periodic cycle.

**PRELIMINARIES.**

**CYCLES, EQUILIBRIA AND CYCLES.** Points \( x \), where \( F(x) = 0 \) are called equilibrium points for the differential equation \( \frac{d}{dt} x = F(x) \). If a solution starts at an equilibrium point, it stays at the equilibrium point for ever. If \( x(t) \) is a solution curve and \( x(T) = x(T') \) for some \( T > 0 \), then the curve that we do not include equilibrium points in this definition. The minimal time \( T \) for which \( x(T) = x(0) \) is called the period of the cycle.

**TRANSVERSE CURVES.** A smooth curve \( y(x) \in R^2 \) is called transverse to the vector field \( x \to F(x) \) at every point \( x \), the vector \( F(x) \) at \( x \) is not tangent to the curve \( y \) passing through \( x \).

**OMEGA LIMIT SET.** The omega limit set \( \omega(x_0) \) of an orbit \( x(t) \) passing through \( x_0 \) is the set of points \( x \), for which there exist a sequence of times \( t_n \) such that \( x(t_n) \) converges to \( x \). Equivalent is the mathematical statement: \( \omega(x_0) = \bigcup_{t \geq 0} \{ x(t) | t \geq 0 \} \).

**JORDAN CURVE THEOREM.** A Jordan curve is a simple closed curve in the plane. "Simple" means that the curve should not have self-intersections or be tangent to itself at any point. The Jordan curve theorem asserts that the curve divides the plane into two disjoint regions, the "inside" and the "outside".

**EXAMPLE OF LIMIT CYCLE.** The differential equation given in polar coordinates as

\[
\frac{d}{dt} r = (1 - r^2), \quad \frac{d}{dt} \theta = 1
\]

is with \( r = \cos(\theta), y = \sin(\theta) \) equivalent to

\[
\frac{d}{dt} r = \cos(\theta) - r \sin(\theta) = (1 - r^2) r - y
\]

\[
\frac{d}{dt} \theta = \theta \cos(\theta) + \gamma \sin(\theta) = (1 - r^2) \theta + y
\]

In this example, all initial conditions away from the origin will converge to the limit cycle.

**EXAMPLE OF ATTRACTION POINT.** The differential equation given in polar coordinates as

\[
\frac{d}{dt} r = r(t^2 - 1), \quad \frac{d}{dt} \theta = 1
\]

is with \( r = \cos(\theta), y = \sin(\theta) \) equivalent to

\[
\frac{d}{dt} r = \cos(\theta) - r \sin(\theta) = (1 - r^2) r - y
\]

\[
\frac{d}{dt} \theta = \theta \cos(\theta) + \gamma \sin(\theta) = (1 - r^2) \theta + y
\]

In this example, all initial conditions away from the limit cycle will converge to the origin or to infinity.

**PROOF OF THE POINCARE-BENDIXON THEOREM.** The aim is to show that if the omega limit set \( \omega(x_0) \) is nonempty, then it either contains an equilibrium or a closed periodic orbit.

(i) There are no equilibrium points on a transverse curve. The vector field \( J \) can therefore not reverse direction along the curve.

(ii) Let \( y \) be a transverse curve. If a solution \( x(t) \) crosses \( y \) more than once, the successive crossing points form a monotonic sequence on the arc \( y \).

Proof. Denote by \( y(t) = y(x(t)), x(t) \) the first two crossing times. We can assume that \( x_1 \geq x_2 \), because if this does not hold, one can reinterpret \( y \) by \( \psi^{-1} \). If \( x_1 < x_2 \), the union of the two smooth arcs \( \{ y(t) | 1 \leq t \leq x_1 \} \) and \( \{ y(t) | x_1 \leq t \leq x_2 \} \) is a closed piecewise smooth curve. By Jordan’s curve theorem, such a curve divides the plane into two different regions. For \( t > x_2 \), the solution \( x(t) \) stays in one of these regions. For the first crossing, \( x(t) = x(0) \) one has therefore \( x_2 > x_1 \).

(iii) It follows from (ii) that no more than one point of any transverse arc \( y \) can belong to the \( \omega \)-limit set \( \omega(x_0) \).

(iv) Given \( y \in \omega(x_0) \). Because a solution \( x(t) \) with \( x(0) \) is generic, in a bounded region, the solution \( x(t) \) is by the existence theorem for differential equations defined for all times. It stays in \( \omega(x_0) \) because \( x(t) \) is invariant under the flow. Assume, there exists no stationary point in \( \omega(x_0) \). Then exists a transverse arc \( y \) passing through \( x_0 \). Because \( \omega(x_0) \) can have only one intersection and \( x(t) \) returns arbitrary close to \( x_0 \), the orbit \( x(t) \) through \( x_0 \) is a periodic orbit.

**DIFFERENT SURFACES.** Does an analogue of Poincaré-Bendixon hold also on other two dimensional spaces? The answer depends on the surface. On the sphere, the answer is yes, on the torus, there are solutions which are neither asymptopic to a limit cycle or equilibrium point. An example of such a curve is in \((u, v) \) and \( u \) which is a solution of the differential equation

\[
\frac{d}{dt} u = 1, \quad \frac{d}{dt} v = 0
\]

Differential equations of the form

\[
\frac{d}{dt} x = F(x, y) \quad \frac{d}{dt} y = G(x, y)
\]

can even show some kind of mixing. You explore the question a bit in a homework problem.
THE POINCARE-BENDIXON THEOREM

Math118, O. Knill

ABSTRACT: The Poincaré-Bendixon theorem tells that the late of any bounded solution of a differential equation in the to convergence either to an attractive fixed point or to a limit cycle. This theorem rules out "chaos" for differential equations in the plane.

THEOREM (Poincaré-Bendixon) Given a differential equation \( \dot{x} = F(x) \) in the plane. Assume \( x(t) \) is a solution curve which stays in a bounded region. Then either \( x(t) \) converges for \( t \to \infty \) to an equilibrium point where \( F(x) = 0 \), or it converges to a single periodic cycle.

PRELIMINARIES

CYCLES, EQUILIBRIA AND CYCLES. Points \( x \), where \( F(x) = 0 \) are called equilibrium points for the differential equation \( \dot{x} = F(x) \). If a solution starts at an equilibrium point, it stays at the equilibrium point for ever. If \( F(x) \) is a solution curve and \( x(t) + \gamma = x(t) \) for some \( \gamma > 0 \), then the curve is called a cycle. Note that we do not include equilibrium points in this definition. The minimal time \( T \) for which \( x(t + T) = x(T) \) is called the period of the cycle.

TRANSVERSE CURVES: A smooth curve \( \gamma(x) \in \mathbb{R}^2 \) is called transverse to the vector field \( x \to F(x) \) if at every point \( x \in \gamma \), the vector \( F(x) \) and at least one tangent vector to \( \gamma \) passing through \( x \) are linearly independent.

OMEGA LIMIT SET: The omega limit set \( \omega^+(x) \) of an orbit \( x(t) \) passing through \( x_0 \) is the set of points \( x \) for which there exists a sequence of times \( t_n \) such that \( x(t_n) \to x \) as \( n \to \infty \). It is the closure of the set \( A \). If \( x \) is a limit point of an orbit \( A \), it is called a limit cycle.

JORDAN CURVE THEOREM: A Jordan curve is a simple closed curve in the plane. "Simple" means that the curve should not have self-intersections or be tangent to itself at any point. The Jordan curve theorem assures that such a curve divides the plane into two disjoint regions, the "inside" and the "outside." This seemingly elementary fact is surprisingly hard to prove.

EXAMPLE OF LIMIT CYCLE. The differential equation given in polar coordinates

\[
\begin{align*}
\frac{dx}{dt} &= r(1 - r^2), \\
\frac{dr}{dt} &= 1
\end{align*}
\]

is with \( x = r \cos(\theta), \ y = r \sin(\theta) \) equivalent to

\[
\begin{align*}
\frac{dx}{dt} &= r \cos(\theta) - r \sin(\theta) \\
\frac{dy}{dt} &= r \sin(\theta) + r \cos(\theta)
\end{align*}
\]

\[
(1 - (x^2 + y^2))x = y \]

In this example, all initial conditions away from the origin will converge to the limit cycle.

EXAMPLE OF ATTRACTIVE POINT. The differential equation given in polar coordinates

\[
\begin{align*}
\frac{dr}{dt} &= r^2 - 1, \\
\frac{d\theta}{dt} &= 1
\end{align*}
\]

is with \( x = r \cos(\theta), \ y = r \sin(\theta) \) equivalent to

\[
\begin{align*}
\frac{dx}{dt} &= r \cos(\theta) - r \sin(\theta) \\
\frac{dy}{dt} &= r \sin(\theta) + r \cos(\theta)
\end{align*}
\]

\[
(1 + x^2 + y^2) - 1)z - y = 0
\]

In this example, all initial conditions away from the limit cycle will converge to the origin or to infinity.

PROOF OF THE POINCARE-BENDIXON THEOREM: The aim is to show that if the omega limit set \( \omega^+(x) \) is nonempty, then it either an equilibrium point or a closed periodic orbit.

(i) There are no equilibrium points on a transverse curve. The vector field \( f \) can therefore not reverse direction along the curve.

(ii) Let \( \gamma \) be a transverse curve. If a solution \( x(t) \) crosses \( \gamma \) more than once, the successive crossing points form a monotonic sequence on the arc \( \gamma \).

The proof of the Jordan curve theorem states that a curve divides the plane into two different regions. For \( t > t_0 \), the solution \( x(t) \) stays in one of these regions. For the next crossing \( x(t_1) = x(t_2) \) one has therefore \( t_1 < t_2 \).

(iii) It follows from (ii) that no more than one point of any transverse arc \( \gamma \) can belong to the limit set \( \omega^+(x) \).

(iv) Given \( y_0 \in \omega^+(x) \). Because a solution \( y(t) \) with \( y(0) = y_0 \) stays by assumption in a bounded region, the solution \( y(t) \) by the existence theorem for differential equations passes for all times all stays in \( \omega^+(x) \).

There exists then a transverse arc \( \gamma \) passing through \( y_0 \). Because \( \omega^+(x) \) \( \gamma \) can have only one intersection and \( y(t) \) returns arbitrary close to \( y_0 \), the orbit \( (x) \) through \( y_0 \) is a single periodic orbit.

DIFFERENTIABLE SURFACES. Does an analog of Poincaré-Bendixon hold also on other two dimensional spaces? The answer depends on the space. On the sphere, the answer is yes, on the torus, there are solutions which are neither asymptotic to a limit cycle or equilibrium point. An example of such a curve is (t, t) and it is a solution of the di erential equation

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \alpha
\]

Differential equations of the form

\[
\frac{d}{dt} \gamma = F(x, \gamma), \quad \frac{d}{dt} \phi = \alpha F(x, \phi)
\]

can even show some weak type of mixing. You explore the question a bit in a homework problem.
2D Linear Systems

\[ y \rightarrow z = \int z \]

1. \((\lambda, \phi)\)
2. \((\mu, \nu)\)
3. \((\beta, \gamma)\)

\[ \text{Volker} \]

\[ x = ax - byx \\
y = cy + bxy \]

Eq. 1:
\[ a_0 x (a - b) y = 0 \\
y (c + b) x = 0 \]

\[ S = \begin{pmatrix} a - by & -bx \\ by & -c + bx \end{pmatrix} \]

\[ \lambda^2 + ac = 0 \quad \lambda = 2i \sqrt{ac} \]

\[ (0, 0) \]

\[ (0, a) \]

\[ \left( \frac{a}{b}, \frac{c}{b} \right) \]

\[ \text{Saddle} \]

\[ \text{Saddle} \]

\[ \left( 0, \frac{c}{2} \right) \]
\[ x_0 = (\alpha) \]

\[
\begin{pmatrix}
\alpha e^{\lambda t} \\
\beta e^{\lambda t}
\end{pmatrix}
\]

\[ \lambda e^t \]

\[ c_1 + c_1 e^t \]

\[ A v_1 = \lambda v_1 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ A v_2 = \lambda v_2 \]

\[ b = 0 \]

\[ \alpha = 0 \]

\[ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ e^{\lambda t} \end{bmatrix} = \begin{bmatrix} A t e^{\lambda t} + e^{\lambda t} \\ \lambda e^{\lambda t} \end{bmatrix} \]
Central Limit Theorem

\[ x(c) = \int e^{i cx} \, dx(x) \]

\[ p(c) = \int e^{-c x} \, d\beta(x) \]

Fisher's z-Score and class

Fisher transformation and z-Score class

\[ f(z) = \int q(y) e^{-iyz} \, dy \]

\[ \int f(c) \, dx(x) = \int f(q(y)) e^{-iyz} dy \, dx(x) \]

\[ = \int q(y) e^{-iyz} \, dx(x) \, dy \]

\[ = \int q(y) \, 0 \, dy \]

Hence

\[ \sum_{i=1}^{n} \text{some term} \]

Use LLN's to prove

\[ \text{some consequence} \]

Theorem on [1.17]

P(\{0, 1\})

Take \( x_i \) to be unbiased

\[ P(x_i = 0) = p \]

\[ P(x_i = 1) = 1 - p \]

\[ E[\{x_i \}] = \sum_{i} f(x_i) p(x_i) \]

\[ P(\{x_i \}) = \prod_{i=1}^{n} P(x_i) \]

\[ E[\{x_i \}] = E[\{x_i \}] = p(p) \]
You have the sequence \( X_n \) of random variables with \( X_n \in \mathbb{F}_n \) and \( E(X_n) < \infty \). Then,

\[
E(X_n | F_n) = X_n
\]

and

\[
P \left( \bigcup_{n=1}^{\infty} \{|X_n| > \epsilon_n\} \right) \leq \frac{E(X_1)}{\epsilon}
\]

We can apply the Borel-Cantelli Lemma. Let

\[
\epsilon_n = \frac{1}{n}
\]

where \( \epsilon_n \to 0 \) as \( n \to \infty \). Then

\[
\sum_{n=1}^{\infty} \epsilon_n = \infty
\]

so

\[
\sum_{n=1}^{\infty} P(|X_n| > \epsilon_n) = \infty
\]

Now, consider

\[
X_n = \sum_{j=1}^{n} Y_j
\]

where

\[
Y_j = X_j - X_{j-1}
\]

Then

\[
E(Y_j | F_{j-1}) = 0
\]

so

\[
E(Y_j) = 0
\]

and

\[
E(Y_j^2) = E(X_j - X_{j-1})^2 = E(X_j^2) - 2E(X_j X_{j-1}) + E(X_{j-1}^2)
\]

Thus

\[
E(Y_j^2) = E(X_j^2)
\]

Therefore,

\[
\sum_{j=1}^{n} E(Y_j^2) = \sum_{j=1}^{n} E(X_j^2)
\]

and

\[
E \left( \sum_{j=1}^{n} Y_j^2 \right) = E \left( \sum_{j=1}^{n} X_j^2 \right)
\]

Thus,

\[
\sum_{j=1}^{n} E(Y_j^2) \to E(X_n^2)
\]

as \( n \to \infty \).