Fourier Series

\[ s_n(x) = \sum_{j=-n}^{n} c_j e^{i j x} = \sum_{j=-n}^{n} \left( \frac{1}{2\pi} \int_{a}^{b} f(x-e) e^{i j x} \, dx \right) e^{i j x} = \frac{1}{\pi} \int f(x) D_n(x) \, dx \]

Dirichlet Kernel

\[ D_n(x) = \sum_{j=-n}^{n} e^{i j x} \]

1. Lebesgue constants
   \[ L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(x)| \, dx = 2 \ln(n) \]

2. \( D_n(x) \) is real-valued, continuous, \( 2\pi \)-periodic for \( n \geq 0 \), positive and non-negative valued.

3. \( D_n \) is even.

4. \( D_n(x) = \frac{\sin((n+1)x)}{2 \sin x} \)

5. \( D_n \) is odd if \( n \) is odd.

6. For each \( n \), \( |D_n(x)| \leq 1 \) for each \( x \in \mathbb{R} \)

7. For each \( n \), \( \int |D_n(x)| \leq \frac{\pi}{2n+1} \)

Fejer Kernel

\[ \sigma_n(f) = \frac{1}{n+1} \sum_{j=-n}^{n} s_j(f) = \frac{1}{n+1} s_{n+1}(f) \]

\[ \sigma_n(f) = \frac{1}{n+1} \int f(x) K_n(x) \, dx = \frac{1}{n+1} \int f(x) \sum_{j=-n}^{n} \delta_j(x) \, dx \]

1. \( K_n(x) \) is real-valued, non-negative (continuous)

2. Even Function

3. For each \( n \), \( \frac{1}{n+1} \int_{-\pi}^{\pi} K_n(x) \, dx = \frac{\pi}{n+1} \int_{-\pi}^{\pi} s_n(x) \, dx = 1 \)

4. For each \( n \), \( K_n \) is symmetric

5. For each \( n \), \( K_n(x) = \frac{1}{2\sin(x/2)} \left( \frac{\sin((n+1)x/2)}{\sin(x/2)} \right) \)

6. For each \( n \), \( 0 \leq K_n(x) \leq \frac{\pi}{(n+1)^2} \)
Convergence of Cesàro means

Thus \( f \in L^1(\mathbb{T}) \), \( \hat{f}(x) = \hat{f}(x) \) exist and

\[
\lim_{n \to \infty} \sigma_n(x) = \frac{1}{2} \left[ f(x) + f(x + \pi) \right].
\]

Thus (non locally) \( f \in L^1(\mathbb{T}) \) if \( \lim_{n \to \infty} \sigma_n(f(x)) = f(x) \) at every Lebesgue point of \( f \).

Then a.e., \( 2\pi \)-periodic as \( \lim_{n \to \infty} \sigma_n(f(x)) = f(x) \) uniformly.

Fourier coefficients

The Fourier coefficients of \( f \in L^1(\mathbb{T}) \) need not converge.

\[
\text{Lusin - Lebesgue: } \quad \lim_{n \to \infty} \sum_{j=-n}^{n} \hat{f}(j) = 0,
\]

or

\[
\sum_{j=-n}^{n} |\hat{f}(j)| \leq M_n.
\]

Thus mapping \( f \mapsto \hat{f} \) from \( L^1(\mathbb{T}) \) into \( c_0(\mathbb{Z}) \) is bounded but not onto.

Proof: Use a specific mapping principle.

Approximation

Thus \( f \in L^p(\mathbb{T}) , \quad 1 < p < \infty \)

\[
\lim_{n \to \infty} \|f - \sigma_n(f)\|_p = 0.
\]
Idea of measure → $\sigma$-algebra → measurable spaces, finite, complete, non-negative, countable, disjoint sets → Borel σ-algebra

Borel measure on $\mathbb{R}^n$

Loeb measure in opposite direction: $\mathcal{F}$ increasing, right continuous

Loeb measure - Strichartz measure

Integration

Measurable Functions

Lebesgue vs. Borel

Composition $f \circ g$ needs both measures to be finite

Measurable Functions

Simple functions: finite, non-negative → standard representation

Non-negative functions: $\int_{\mathbb{R}^n} f \, d\mu$ non-negative → standard representation

Fatou's Lemma

Convergence: uniform, pointwise, etc., in measure $\mathcal{F}_\sigma$ Measurable

Produce measure
Real Analysis

Proof 1

\[ \sum \binom{n}{k}^2 \binom{2n}{2k} = 1 \]

\[ \sum \binom{n}{k}^2 \binom{2n}{2k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n+1} \]

\[ \sum \binom{n}{k}^2 \binom{2n}{2k} = \sum \binom{n}{k}^2 \binom{n}{k} \]

\[ \sum \binom{n}{k}^2 \binom{2n}{2k} = \sum \binom{n}{k}^2 \binom{n}{k} = 2^{2n} \binom{n}{k} \]

\[ \sum \binom{n}{k}^2 \binom{2n}{2k} = \frac{2}{2n+1} \sum \binom{n}{k} \binom{2n}{2k} \]

Proof 2

Let \( k \in \mathbb{N} \) and \( k_n \in \mathbb{N} \) be sequences such that \( n \to \infty \) implies \( k_n 

For \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \), let \( f : \mathbb{N} \to \mathbb{N} \)

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{2k} \]

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{2k} = 2^{2n} \binom{n}{k} \]

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{2k} = \frac{2}{2n+1} \sum_{k=0}^{n} \binom{n}{k} \binom{2n}{2k} \]

Note: These can be approximated by poly. 

Note: \( f \) can grow to \( \binom{n}{k} \) as \( n \to \infty \), so \( \binom{n}{k} \) is approximately \( \binom{2n}{2k} \) for \( n \) large enough.
Proof

1. Suppose \( \varphi(x) \in \mathcal{C}(X, Y) \) is a uniformly continuous function. Let \( \epsilon > 0 \). For each \( x \), \( |\varphi(x) - \varphi(y)| < \epsilon \) whenever \( \|x - y\| < \delta \). Let \( \delta_0 = \delta \).

2. For \( \varphi \) to be uniformly continuous, define \( \delta \) as follows:

\[ \|x - y\| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon \]

where \( \epsilon > 0 \) is arbitrary. Choose \( \delta \) such that

\[ |\varphi(x) - \varphi(y)| < \epsilon \quad \text{whenever} \quad \|x - y\| < \delta \]

in a neighborhood of \( x \). Hence, \( \varphi \) is uniformly continuous.

Observation:

\( \varphi \) is uniformly continuous if and only if \( \varphi \) is a uniform limit of uniformly continuous \( \varphi_n \).

Conclusion:

If \( \varphi \) is uniformly continuous, then \( \varphi \) is continuous.

\[ \lim_{n \to \infty} \varphi_n(x) = \varphi(x) \quad \text{uniformly} \]

for each \( x \) in \( X \).
**Convex Functions**

**Definition:** A function $f$ is convex if $f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$ for all $x, y$ in the domain of $f$ and all $a \in [0, 1]$.

**Examples:**
- **Line segment:** $f(c) = f(x) + \lambda (y-x)$
- **Parabola:** $f(c) = f(x) + \lambda (y-x)$

**Properties:**
- **Function values:** $f(x) \leq f(c) \leq f(y)$
- **Function derivatives:** $f'(c)$ is a weighted average of $f'(x)$ and $f'(y)$.

**Convexity Tests:**
- **Second derivative:** $f''(x) \geq 0$ for all $x$.
- **Subdifferential:** $\partial f(c)$ contains $0$.

**Applications:**
- **Economic theory:** Convex functions are used in optimizing problems.
- **Optimization:** Convex optimization problems have unique solutions.

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**Continuity & Differentiability**

**Continuity:**
- $f$ is continuous at $c$ if $\lim_{x \to c} f(x) = f(c)$.

**Differentiability:**
- $f$ is differentiable at $c$ if $f'(c)$ exists.
- $f$ is differentiable on an interval if it is differentiable at every point in the interval.

**Lipschitz Continuity:**
- A function $f$ is Lipschitz continuous on a set $S$ if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in S$.

**Examples:**
- **Linear functions:** $f(x) = ax + b$ are Lipschitz continuous with $L = |a|$.
- **Absolute value function:** $f(x) = |x|$ is Lipschitz continuous with $L = 1$.

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Measures

A C(X) algebra \( \mu : A \to [0, \infty] \) is measureable if:

1. \( \mu(\emptyset) = 0 \)
2. \( \mu(A \cup B) = \mu(A) + \mu(B) \) if \( A \cap B = \emptyset \)
3. \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \)

Outer Measure

On a measurable space \( X \) is a finite \( \mu : P(X) \to [0, \infty] \) s.t.

1. \( \mu(\emptyset) = 0 \)
2. \( \mu(A) \leq \mu(B) \) if \( A \subseteq B \)
3. \( \mu(B) = \sum_{i=1}^{\infty} \mu(B_i) \) if \( B = \bigcup_{i=1}^{\infty} B_i \)

Examples of C(X) algebra

Family of Borel measurable functions \( \mathcal{E} \)

\( \mathcal{E} \) is a family of additive sets.

A C(X) algebra \( A \) is a measure algebra if \( \mu \) and \( \gamma \) are measures.

Characterization of measure algebra

A C(X) algebra \( A \) is a measure algebra if it is complete and separable.

Closed under \( \gamma \)-additivity.

Closed under \( \mu \)-additivity.

Only \( \mathcal{E} \) \( \gamma \)-measurable if \( \mu(A) = \gamma(A) \) for all \( A \in \mathcal{E} \).

Outer measures \( \gamma \) and measures \( \mu \) are related by:

\[ \gamma(A) = \inf \{ \mu(B) : B \supseteq A \} \]

Corollary

If \( \mu \) is a measure on \( X \), then \( \gamma \) is a measure on \( X \) if \( \gamma(\emptyset) = 0 \) and for all \( A \in \mathcal{E} \),

\[ \gamma(A) = \inf \{ \mu(B) : B \supseteq A \} \]

Measure \( \gamma : \mathcal{E} \to [0, \infty] \) is additive:

\[ \gamma(A \cup B) = \gamma(A) + \gamma(B) \] for \( A \cap B = \emptyset \)

Additive measure \( \gamma \) on \( \mathcal{E} \) is a measure:

\[ \gamma(A) = \sum_{i=1}^{\infty} \gamma(A_i) \] for \( A = \bigcup_{i=1}^{\infty} A_i \)

Theorem

A C(X) algebra \( A \) is a measure algebra if it is complete and separable.

By completeness, \( \gamma \)-measurable functions \( f \) are \( \mu \)-measurable.

For any measure \( \mu \) on \( A \), \( \mu \) is \( \gamma \)-measurable.

V. \( \gamma \)-measurable functions \( f \) satisfy \( \gamma(f) = \mu(E) \) for \( E \in \mathcal{E} \) and \( f \) is \( \mu \)-measurable.

If \( \mu \) is a measure on \( A \), then \( \gamma \) is the unique extension of \( \mu \) to a measure on \( A \).
Preliminary Theorems

Weak convergence theorem: If \( X_n \to X \) in distribution, \( n \to \infty \), then
\[
\mathbb{E} f(X_n) \to \mathbb{E} f(X)
\]
for every bounded and continuous function \( f \).

Mann iteration converges
\[
\text{If } 
\begin{align*}
\alpha_n &\geq 0, \\
\sum_{n=0}^{\infty} \alpha_n &< \infty,
\end{align*}
\text{ then } x_{n+1} = \beta x_n + (1-\beta) f(x_n)
\]
for some \( \beta \in (0,1) \).

Fejér's Lemma: \( \sum_{n=0}^{N-1} f(n) \leq \frac{1}{N} \sum_{n=0}^{N-1} f(n) \)

Dominated convergence

Lemma: \( a \geq 0, b \geq 0 \), \( c \geq 0 \), \( a+b+c \leq a + b + c \)

Holder's inequality: \( \| f \|_p \leq \| f \|_r \leq \| f \|_1 \) for \( p, r \) such that \( \frac{1}{p} + \frac{1}{r} = 1 \).

Lemma: \( a = |f(x)|^2, \quad b = |g(x)|^2 \).

Weak convergence: \( f_n(x) \to f(x), \quad g_n(x) \to g(x) \Rightarrow \| f_n - f \|_p \to 0 \) for \( p \in [1,\infty] \).
\[ F_{EAC} = f(0) - f(0) - \int_{b}^{a} f'(x) \, dx \]

**Note:**
1. \( F_{EAC} = f_a \)
2. Every functional integral is AC
3. \( F_{EAC} = 0 \) \( F_{EAC} = f_{a} \) \( \frac{d}{dx} f_{a} = C_{EAC} \)
4. \( f_{EAC} = f_{EAC} \)
5. \( f_{EAC}, a \Rightarrow f_{EAC} \)
6. Space derivative \( \rightarrow \) Galip

\[ f_{EAC} = \frac{d}{dx} f_{a} \]

\[ x \in [a, b] \]

**Every AC is the absolute integral of its derivative**

\( F_{EAC} \) is an absolute integral \( \iff \) \( F_{EAC} \) is a constant

**Proof:**
- \( F_{EAC} \) by Prop 4.14
- \( F_{EAC} = f_{EAC} - f_{EAC} + \int_{a}^{b} f'(x) \, dx \) \( \text{Galois integral} \)
- \( f_{EAC} \) is a.e. \( \Rightarrow \) \( f_{EAC} \) is a.e. \( \text{Galois integral} \)
- \( \int_{a}^{b} f'(x) \, dx + f_{EAC} = f_{EAC} \)
- \( \int_{a}^{b} f'(x) \, dx \)
- \( f_{EAC} = f_{a} \)
- \( f_{EAC} = f_{a} \) G.L.
- \( \text{Lemma} \) \( f_{EAC} = \int f'(x) \, dx \)
- \( f_{EAC} = f_{EAC} - f_{EAC} + \int_{a}^{b} f'(x) \, dx \) \( \text{Galois integral} \)
- \( f_{EAC} \) is a.e. \( \Rightarrow \) \( f_{EAC} \) is a.e. \( \text{Galois integral} \)
- \( f_{EAC} = f_{a} \) constant
- \( \Rightarrow \) \( f_{EAC} = f_{a} \) is a.e. \( \text{Galois integral} \)
Convergence means many things.

NUS, norm, norm, Equivalence of norms

Banach: NUS complete ⇒ every absolutely convergent series converges

\[ \overline{\text{closed}} \quad \overline{\text{continuous}} \quad \overline{\text{continuous}} \]

\[ Y \text{ complete} \Rightarrow (X,Y) \text{ complete} \]

**FUNCTIONS**

中小企业 \( \Rightarrow \) Non-Banach Theorem \( \Rightarrow \) shows \( \Rightarrow \) using \( \Rightarrow \) and \( \Rightarrow \)

Prove \( \forall x \in X, \lim_{n \to \infty} f(x_n) = f(x) \Rightarrow \exists f \in X^* \to Y \text{ in } M_{X,Y} \]

Riemann Lemma.

**Basic Category**

- Complete metric space
  - \( \Rightarrow \) Hausdorff space
  - \( \Rightarrow \) discrete \( \Rightarrow \)
  - \( \Rightarrow \) metric space

**Open Mapping**
- \( T \) is open \( \Rightarrow \) \( T^{-1}(U) \) open
- \( T \) is closed \( \Rightarrow \) \( T^{-1}(U) \) closed

**Closed Graph**
- \( T \) is closed \( \Rightarrow \) \( T^{-1}(U) \) closed

**Linear Algebra**
- \( T \) is linear \( \Rightarrow \) \( T^{-1}(U) \) open

**Hilbert Spaces**

Inner Product

\[ \langle x,y \rangle \text{ inner product} \]

Conjugate Symmetry

\[ \langle x,y \rangle = \overline{\langle y,x \rangle} \]

Hilbert space: complete metric space

Banach space: linear space \( \Rightarrow \) complete metric space

Pythagorean Theorem

- \( \text{In a Hilbert space } H = A \perp A^* \)
- \( f(x) = \langle x,y \rangle + \sum_{n=1}^{\infty} \langle x_n,y \rangle \]

Orthogonal projection \( \Rightarrow \) \( f \) is \( \text{in } H \)

F CA \[ \Rightarrow \text{Hilbert space } \Rightarrow \text{Banach space} \]

\[ \text{Orthogonal projection} \]

- \( \langle x,y \rangle = 0 \) \text{ orthogonal}
- \( \langle x,x \rangle = 1 \) \text{ inner product}
- \( \langle x,x \rangle = 0 \) \text{ orthogonal}
- \( \langle y,y \rangle = 1 \) \text{ inner product}
- \( \text{Hilbert space } \Rightarrow \text{Banach space} \)

**Separation**

- \( \langle x,y \rangle = 0 \) \text{ orthogonal}
- \( \langle x,x \rangle = 1 \) \text{ inner product}
- \( \langle y,y \rangle = 1 \) \text{ inner product}