Atmospheric Dynamics

Lecture 5: Quasigeostrophic theory

1 Pressure as a vertical coordinate

The primitive equations can be rewritten in a somewhat more tractable form if we consider pressure as a vertical coordinate. In general pressure drops monotonically with altitude so this amounts usually to just a stretching transformation. The use of the hydrostatic equation and elementary calculus allows us to rewrite the horizontal momentum equations in a somewhat simpler form as

$$\begin{array}{c} \frac{du}{dt} - fv = -\Phi_x \\ \frac{dv}{dt} + fu = -\Phi_y \end{array} \tag{1}$$

where the field $\Phi \equiv gz$ is called the geopotential and the total derivative becomes

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

where $\omega = \frac{dp}{dt}$ is the pressure coordinate vertical velocity. The hydrostatic equation can easily be transformed to

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \tag{2}$$

Rederivation of the continuity equation in pressure coordinates (Holton section 3.1) results in a considerable simplification to a solenoidal form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{3}$$

Finally the first law of thermodynamics can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)T - S_p\omega = \frac{Q}{c_p} \tag{4}$$

where the parameter

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$$

is called the static stability parameter which combines the mechanical energy term and the vertical advection of temperature. It typically has a value of $5 \times 10^{-4} K(Pa)^{-1}$ in the mid troposphere.

We have a new set of five variables u, v, ω, Φ and T (we are neglecting the effects of humidity for the present) with five equations.

2 Geostrophy and the Rossby number

As was mentioned in the previous lecture, large scale mid latitude (synoptic) variability is characterized by an approximate balance between the pressure gradient term and the Coriolis term in the horizontal momentum equations. The former can be replaced with the geopotential gradient term if we consider the pressure coordinate equations. If we choose a typical wind scale U and horizontal length scale L as well as assume a particular latitude by selecting a Coriolis parameter f_0 then this balance can be expressed through the Rossby number

$$Ro = \frac{U}{f_0 L}$$

having a value much smaller than unity. This follows since the Rossby number gives the ratio in magnitude of the total time derivative and Coriolis terms in the horizontal momentum equations. In pressure coordinates this balanced flow (the geostrophic flow) is given by

$$\begin{aligned} v_g &= \tfrac{1}{f_0}\Phi_x \\ u_g &= -\tfrac{1}{f_0}\Phi_y \end{aligned} \tag{5}$$

In a horizontal sense then this flow is at right angles to the gradient vector of geopotential i.e. it is tangential to the contours of geopotential. It is easily checked that the horizontal divergence of this flow vanishes implying that any such divergence in synoptic flow must arise from a deviation from geostrophic balance. Note that the continuity equation implies that vertical velocity (in pressure coordinates at least) is the vertical integral of horizontal divergence and so must also arise from (small) deviations from geostrophic flow.

3 Quasigeostrophic flow

A useful way to analyze synoptic variability is to split the flow into geostrophic and ageostrophic parts. For small Rossby numbers the latter component is small (the ratio of the magnitudes is approximately Ro) and so we can simplify our equations using a perturbation analysis. Formally we write

$$\vec{u} = \vec{u}_g + \vec{u}_a$$

insert this decomposition into equations (1), (2), (3) and (4) and retain only terms of first order in Ro (the geostrophic terms of order unity drop out). It is common also to assume that the ratio of the variation in the Coriolis parameter to the mean value f_0 is also of the same order as Ro. This assumption is known as the <u>beta plane approximation</u> and allows us to write (by linearization)

$$f = f_0 + \beta y$$

and assume that

$$\frac{|\beta y|}{|f_0|} \sim Ro$$

The momentum equations then become

$$\begin{array}{c} \frac{d_g u_g}{dt} - f_0 v_a - \beta y v_g = 0 \\ \frac{d_g v_g}{dt} + f_0 u_a + \beta y u_g = 0 \end{array} \tag{6}$$

where

$$\frac{d_g}{dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

Note that the vertical advection term has vanished because the geostrophic component of the vertical velocity is zero. Note also that the geopotential has vanished due to the removal of the geostrophic balance terms. Every term in equation (6) has magnitude of order Ro. Since the geostrophic flow is non-divergent the continuity equation becomes

$$\nabla.\overrightarrow{u}_a + \frac{\partial \omega}{\partial p} = 0 \tag{7}$$

which shows that vertical velocities associated with synoptic flows derive from the ageostrophic flow. The hydrostatic equation remains the same as equation (2) and becomes simply a diagnostic equation for the geopotential. Finally the temperature equation simplifies to

$$\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) T - \overline{S}_p \omega = \frac{Q}{c_p} \tag{8}$$

where the over bar means a horizontal average (we are assuming horizontal variations in temperature are small relative to vertical ones). We are assuming

$$T_{tot}(x,y,p,t) = T_0(p) + T(x,y,p,t)$$

where $T_0(p)$ is the horizontal mean temperature.

Equations (5), (6), (7), (2) and (8) constitute a solvable set of equations in the variables $\overrightarrow{u}_a, \overrightarrow{u}_g, \omega, \Phi$ and T and are known as the equations for the quasigeostrophic flow. They are fundamental to the dynamical analysis of the synoptic flow.

4 Vorticity

The vertical component of vorticity (we shall call this simply vorticity) associated with the geostrophic flow in the quasigeostrophic approximation is a particularly useful dynamical variable. It is easily related to geopotential by differentiating the first equation of (5) by x and subtracting the y derivative of the second

$$\varsigma_g = \frac{1}{f_0} \nabla^2 \Phi$$

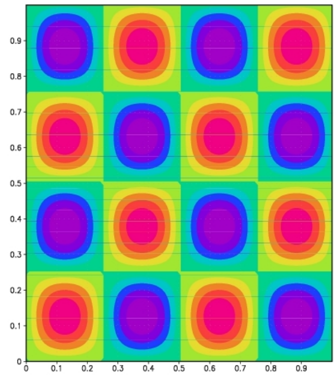

Figure 1: Idealised geopotential pattern

which implies a particularly simple relation between geostrophic vorticity and geopotential which reverses when a switch is made between hemispheres. Consider a simple geopotential pattern

$$\Phi = \sin(kx)\sin(ly)$$

which is displayed in Figure 1 for $k = l = 8\pi$. The geostrophic flow is along the contours of geopotential and is easily shown in the Northern hemisphere to be clockwise around the geopotential minima (lows) and conversely around the maxima (highs). In the Southern hemisphere the direction reverses because the Coriolis parameter reverses sign. The vorticity is given by

$$\varsigma_g = -\frac{1}{f_0}(k^2 + l^2)\sin(kx)\sin(ly)$$

i.e. overlaying the geopotential pattern but with the reverse sign. The greater the wavenumbers k and l the more rapid the circulation around the vortices.

A prognostic equation for the geostrophic vorticity is obtained by a similar manipulation using equations (6):

$$\frac{d_g\varsigma_g}{dt} = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g$$

where we have used the fact that the (horizontal) divergence of the gesotrophic flow is zero. This may be easily rearranged to read

$$\frac{\partial\left(\varsigma_g + f\right)}{\partial t} = -\overrightarrow{u}_g.\nabla\left(\varsigma_g + f\right) + f_0\frac{\partial\omega}{\partial p}$$

This is the vorticity equation for quasi-geostrophic flow and is to be compared with more general equations from the third lecture. Note that the total vorticity $\varsigma_g + f$ can change as a result of the vertical stretching of vortex tubes (third term) or as a consequence of the horizontal advection of relative and planetary vorticity (second term).