Atmospheric Dynamics

Lecture 6: Linearization Part 1

1 General concepts

The primitive equations are sufficiently non-linear that analytical solutions are very difficult to obtain except in special cases. A general approach to this has been the extensive use of numerical models. This is clearly however not a general theoretical approach and has rather the flavour of experimental physics. A common approach to this difficulty has been linearization about a variety of mean states. The solutions of the resulting multi-component equations are wave-like disturbances which may grow, decay or remain at constant amplitude depending on what the mean state is and what dissipation exists in the linearized equations. When the disturbances grow we refer to the analysis as linear instability theory while in the other case one often refers to the study as linear wave analysis. A particularly simple and revealing starting point for linearization analysis is the case that the mean state is at rest.

2 Linearization about a state of rest

Particularly useful approximate solutions of the primitive equations can be obtained by linearizing them about a state of rest and assuming that the background (mean) vertical density structure is horizontally but not vertically uniform. In order to simplify the presentation we shall assume incompressibility and hydrostatic equilibrium for the present but later consider how the general case pans out. The momentum equations for small perturbations about the state of rest are

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ g\rho &= -\frac{\partial p}{\partial z} \end{aligned} \tag{1}$$

where we are using the abbreviations x to indicate distance along circles of latitude (called the zonal coordinate), y to indicate distance along longitude circles (called the meridional coordinate) and z is the vertical coordinate which is perpendicular to geopotential surfaces. The Coriolis parameter f is $2\Omega\sin\varphi$ and $\rho_0 = \rho_0(z)$ is the density profile. The equation of continuity in the incompressible case is

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

If we assume that there is no sources of heat and moisture then the incompressible version of the equation of state implies that $\frac{d\rho}{dt} = 0$ which when linearized becomes

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0 \tag{3}$$

3 Separation of Variables and the vertical Sturm-Liouville equations

These five equations have the five unknowns (u, v, w, p, ρ) and general solutions are possible. Those of interest to us are separable in the vertical direction. We choose the following separation for the five variables

$$\begin{aligned} \overrightarrow{u} &= \widehat{u}(z)\overrightarrow{U}(x,y,t) \\ p &= \widehat{p}(z)\eta(x,y,t) \\ \rho &= \widehat{\rho}(z)\upsilon(x,y,t) \\ w &= \widehat{h}(z)\widetilde{w}(x,y,t) \end{aligned}$$

where for reasons that will become clearer later we assume that $\widehat{u}$ and $\widehat{h}$ have the dimensions of length whereas $\widehat{p}$ and $\widehat{\rho}$ have their ordinary units (implying, of course, that the horizontal dependent components are dimensionless). The vertical equations can now be deduced up to a separation constant from the linearized set above. We obtain the four equations for the vertical part of the solutions:

$$\begin{aligned} \rho_0\widehat{u} &= \widehat{p}/g \\ g\widehat{\rho} &= -\widehat{p}_z \\ \widehat{h}_z &= \widehat{u}/H_0 \\ \widehat{\rho} &= (\rho_0)_z\,\widehat{h} \end{aligned} \tag{4}$$

where the factor g has been included in the first equation for dimensional consistency and H_0 is the separation constant which has dimension of length.

Combining the first and third equations and the second and fourth reduces this to

$$\begin{aligned} gH_0\widehat{h}_z &= \widehat{p}/\rho_0 \\ \widehat{p}_z &= -g\left(\rho_0\right)_z \widehat{h} \end{aligned} \tag{5}$$

and this is easily condensed further to

$$\frac{1}{\rho_0}(\rho_0\widehat{h}_z)_z + \frac{N^2}{c_e^2}\widehat{h} = 0 \tag{6}$$

where $N \equiv \left(-g\rho_0^{-1}\left(\rho_0\right)_z\right)^{1/2}$ is termed the Brunt-Vaisala frequency and measures the stability of the background stratification. Note that is always real if the density profile is stable. Parcels of fluid displaced within a density stratification will tend to oscillate at this frequency due to buoyancy effects. The separation constant has been "renamed" $c_e^2 \equiv gH_0$ where c_e is referred to as shallow water speed for reasons that will become clearer later. The reason for assuming that c_e^2 is positive is because the operator $H = \frac{-1}{\rho_0 N^2}\partial_z\rho_0\partial_z$ may easily be shown to be positive providing that the stratification is stable. The lower boundary condition is that the vertical velocity vanishes and so

$$\widehat{h}(0) = 0 \tag{7}$$

Equations (6) and (7) together with the assumption that $N > 0$ form a (semi-infinite) Sturm-Liouville eigensystem. The mathematical literature on such systems is extensive and the eigenvalues c_e^2 can be shown to be positive and the spectrum generally has discrete and continuous parts. The discrete portion is generally of greatest physical interest. The eigenvectors in this system are called the *normal, vertical or baroclinic/barotropic modes.* The mode with the greatest eigenvalue and the simplest (one signed) vertical structure for its eigenvector is the so-called *barotropic* mode which has a shallow water speed of around $200ms^{-1}$. The other modes are called the *baroclinic* modes and have more complex vertical structures. For the observed stratifications they have smaller shallow water speeds (the first baroclinic mode has a typical shallow water speed of around $50ms^{-1}$). Because the system here is Sturm-Liouville, the vertical modes satisfy an orthogonality condition and are complete in the sense that an arbitrary solution may be decomposed into a unique linear combination of vertical modes. Baroclinic modes play an important role in the understanding of tropical dynamics while barotropic modes are of greater importance in the mid-latitudes.

4 The linear shallow water equations

Corresponding to the four equations in the vertical (4) there are a set of equations governing the horizontal flow

$$\begin{array}{lcl} \frac{\partial U}{\partial t} - fV & = & -g\frac{\partial \eta}{\partial x} \\ \frac{\partial V}{\partial t} - fU & = & -g\frac{\partial \eta}{\partial y} \\ \widetilde{w} & = & -H_0(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}) \\ \widetilde{w} & = & \frac{\partial v}{\partial t} = \frac{\partial \eta}{\partial t} \end{array} \tag{8}$$

The last equality here coming from the hydrostatic equation. The third and fourth equations here can be combined and a new variable $h \equiv g\eta$ introduced. The resulting equations

$$\begin{array}{rcl} U_t - fV & = & -h_x \\ V_t + fU & = & -h_y \\ h_t + c_n^2(U_x + V_y) & = & 0 \end{array} \tag{9}$$

are commonly referred to as the linear shallow water equations. We shall consider their solution further in the next Lecture however it is worth examining here the solutions which occur when the Coriolis term is dropped as these have wide applicability in atmospheric flows particularly and illustrate simply some important dynamical properties. The equations in this case are easily reduced to one constant coefficient linear PDE for h. The x derivative of the first equation; the y derivative of the second and the t derivative of the third are all combined to obtain

$$h_{tt} = c_n^2(h_{xx} + h_{yy}) \tag{10}$$

Consider now a general wave like solution of the form

$$h = h_0 \exp(i(\omega t - kx - ly)$$

Substitution into (10) gives the wave dispersion relation

$$\omega^2 = c_n^2(k^2 + l^2)$$

Waves with this dispersion relation (or approximately) are called *gravity waves* and their group velocity is given by

$$\begin{array}{rcl} u_g = \frac{\partial \omega}{\partial k} & = & \frac{c_n^2 k}{\sqrt{k^2+l^2}} \\ v_g = \frac{\partial \omega}{\partial l} & = & \frac{c_n^2 l}{\sqrt{k^2+l^2}} \end{array}$$

which has magnitude equal to the shallow water speed for the particular vertical mode they are derived from. The direction of propagation is preferentially in the direction of greatest wavenumber or smallest wavelength. Gravity waves are often seen in the atmosphere and are generated by rapidly varying forcing such as convective systems and flow over topography.

5 Generalization to a compressible fluid

There is little inherent difficulty in generalizing our incompressible derivation to the compressible case and little of great additional qualitative interest is revealed. It is convenient in this case to work with pressure vertical coordinates and use the equations derived in the first section of the last Lecture. The mathematics of the derivation of the Sturm Liouville system is essentially identical in form to the incompressible case discussed above (exercise for interested students). The only complication concerns boundary conditions. The vertical domain is mapped from semi-infinite to finite by the use of pressure. The boundary condition at $p = 0$ is rather obvious ($\omega = 0$) but the boundary condition at the bottom of the atmosphere is a little less obvious but can be shown to be

$$\omega = \rho_0(p_l)\frac{\partial \Phi}{\partial t}$$

where the maximum pressure is p_l. The finite domain implies a fully discrete spectrum for the positive eigenvalues of the problem and they are related to the incompressible solutions in a rather clear way albeit with modified values. The horizontal equations remain identical i.e. they are the shallow water equations.

6 Other linearizations of the equations

We consider first some basic concepts in dynamical systems theory. A general (unforced) linearized dynamical system may be written as

$$\frac{\partial \psi}{\partial t} = A\psi \tag{11}$$

where the vector ψ specifies the state of the system and the operator A governs the time evolution. The eigenvectors of A are often referred to as the *normal modes* and the corresponding complex eigenvalues determine the growth/decay and oscillatory frequency of these modes. In conservative dynamical systems A satisfies the anti-hermitian property[1]

$$A = -A^*$$

where the star indicates the Hermitian conjugate. This relation implies that it has purely imaginary eigenvalues[2] which means that the normal modes oscillate with constant amplitude. This can be shown also by defining the so-called *propagator* $U(t', t)$ which takes a state vector at a time t and transforms it to the appropriate state vector at time t'. It is easily demonstrated by discretizing (11) and iterating that

$$U(t', t) = \exp((t' - t)A)$$

from which it is easily demonstrated that if A is antihermitian then $U(t', t)$ is orthogonal which implies that it preserves the norm of the state vectors.

The shallow water equations control the dynamics of solutions obtained by linearizing the primitive equations about a state of rest and we can write these in the matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} U/c \\ V/c \\ h/c^2 \end{pmatrix} = \begin{pmatrix} 0 & f & -c\frac{\partial}{\partial x} \\ -f & 0 & -c\frac{\partial}{\partial y} \\ -c\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} & 0 \end{pmatrix}\begin{pmatrix} U/c \\ V/c \\ h/c^2 \end{pmatrix}$$

where for convenience we have "non-dimensionalized" the equations using the shallow water speed. Now as is well known (and can be shown intuitively by discretization) the partial differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are antihermitian and so it follows easily that the operator A for the shallow water equations is also. Thus these equations have purely oscillatory solutions and the system is conservative.

If the equations are linearized about a state of non-rest then, in general, the operator A *is no longer antihermitian.* An obvious example is a linearization about a linearly varying zonal velocity $\overline{U}$. If we assume that this variation occurs in the meridional direction then

$$A = \begin{pmatrix} -\overline{U}\frac{\partial}{\partial x} & f - \overline{U}_y & -c\frac{\partial}{\partial x} \\ -f & -\overline{U}\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} \\ -c\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} & 0 \end{pmatrix}$$

[1] One may need to transform state variables to make this apparent.

[2] As should be well known, Hermitean operators have real eigenvalues

In this case A is obviously not antihermitian and in fact for $\overline{U}_y$ large enough there exist growing normal modes. This is commonly referred to as barotropic instability and occurs frequently in many situations in the atmosphere.