Derivative Securities Final Exam

Fall 2000 – G63.2791 – Professor Kohn

- You may bring two 8.5×11 pages of notes [both sides] to this exam.
- Put your answers on the exam paper; use the back of the page if you need more space, and attach additional sheets if necessary. I will grade *only* your exam paper, not your scratch paper.
- There are 6 "short-answer" problems, worth 10 points each, and 6 "long-answer" problems, worth 20 each, for a total possible score of 180 . Do the questions you find easiest first.
- Show your work, and explain all answers (at least briefly). Partial credit will be given for correct ideas.

NAME: ____________________________

A) Short-answer problems: 10 points each

1. _______
2. _______
3. _______
4. _______
5. _______
6. _______

B) Long-answer problems: 20 points each

1. _______
2. _______
3. _______
4. _______
5. _______
6. _______

Total: ______________

Part A: Short-answer questions. Each of the six questions in part A can be answered in just a few lines.

1. (10 points) True or false: A one-period market with two assets and three distinct final states is necessarily incomplete. Briefly justify your answer.

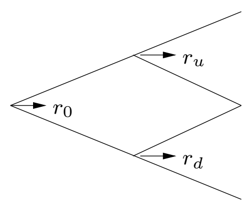
2. (10 points) True or false: An option can be hedged using long and short positions in other options, without ever trading the underlying security. Briefly justify your answer.

3. (10 points) True or false: As one increases the strike price K (keeping all other parameters fixed), the value of a call option increases. Briefly justify your answer.

4. (10 points) Suppose the price process is $ds = r(t)sdt + \sigma(t)sdw$ where $r(t)$ and $\sigma(t)$ vary deterministically. What stochastic differential equation does $\log s(t)$ satisfy?

5. (10 points) Suppose the term structure of interest rates is flat at 5 percent per annum, compounded annually. What is the value of a forward rate agreement which gives its holder the obligation to borrow 1000 dollars a year from now, for a one year term, at 4 percent per anuum?

6. (10 points) Consider the two-period interest rate tree shown below. Assume the risk-

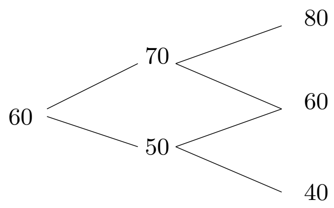


neutral probability is $q = 1/2$. The time interval is δt. Fill in the blanks and explain:

$$B(0, 2\delta t) = \_\_ e^{-(r_0+r_u)\delta t} + \_\_ e^{-(r_0+r_d)\delta t}.$$

Part B. Longer problems. The six problems in Part B call for somewhat longer answers.

1. (20 points) Suppose the price of a non-dividend-paying stock is restricted to the binomial tree shown below. Assume the risk-free rate is $r = 0$. Consider a call option



with strike price 60, maturing at the end of the second time period.

(a) Find the value of this call by working backward in the tree.

(b) Specify the replicating (hedge) portfolio at time $t = 0$.

(c) Suppose the stock goes up to 70 in the first time period. How should the replicating portfolio be changed? Verify that this change is self-financing.

2. (20 points) For a non-dividend-paying stock with lognormal dynamics we know the value of any option is

$$e^{-rT} E_{\mathrm{RN}}[\phi(s_T)]$$

where T is the maturity, r is the risk-free rate, and ϕ is the payoff. We also know that if X is Gaussian with mean m and variance v then

$$E[e^{aX} \text{ restricted to } X \geq k] = e^{am+\frac{1}{2}a^2v} N(d)$$

with $d = (-k + m + av)/\sqrt{v}$. Make appropriate use of these facts to value each of the following:

(a) The option whose payoff is 1 if $s_T > K$ and 0 if $s_T \leq K$.

(b) The option whose payoff is s_T if $s_T > K$ and 0 if $s_T \leq K$.

3. (20 points) Suppose $s(t)$ solves the SDE $ds = rsdt + \sigma sdw$ with $s(0) = s_0$. Let $V(s,t)$ solve the Black-Scholes PDE $V_t + rsV_s + \frac{1}{2}\sigma^2 s^2 V_{ss} - rV = 0$.

   (a) What stochastic differential equation does $e^{-rt}V(s(t),t)$ solve?

   (b) Using your answer to (a), show that $V(s_0, 0) = e^{-rT}E[V(s(T),T)]$.

   (c) Why is this important?

4. (20 points) When we use a binomial tree to value an option on foreign currency, the risk-neutral probability is determined by

$$s_{\text{now}} = e^{-(r-D)\delta t}[q s_{\text{up}} + (1-q) s_{\text{down}}]$$

where s represents the exchange rate, r is the domestic risk-free rate, and D is the foreign-currency risk-free rate. Explain the origin of this formula.

5. (20 points) Some questions about American options:

(a) Explain why early exercise is never optimal, for an American call on a lognormal non-dividend-paying stock.

(b) Show that early exercise can be optimal, for an American put on a lognormal non-dividend-paying stock.

6. (20 points) We studied several applications of Black's formula. In each case, the value of the option has a formula of the form

$$\text{value} = \text{prefactor} \cdot [F_0 N(d_1) - K N(d_2)] \quad \text{or} \quad \text{value} = \text{prefactor} \cdot [K N(-d_2) - F_0 N(-d_1)]$$

with

$$d_1 = \frac{\log(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(F_0/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Please specify which version of Black's formula is appropriate, and say how to choose the prefactor, F_0, and K, for each of the following situations:

(a) A put option on a foreign currency rate, in a constant-interest-rate setting, if the domestic risk-free rate is r and the foreign risk-free rate is D.

(b) A caplet, with cap rate R_{fix} and principal L, for lending during the time interval $t_1 < t_2$.

(c) A swaption which gives its holder the right to enter into a two-year swap a year from now, receiving a specified fixed rate R_{fix} and paying the floating rate. Assume the swaption has notional principal L, and its payments are made annually.