Derivative Securities – Homework 3 – due 10/17/00
Solutions will be distributed 10/24/00

Problems 1-4, drawn from Jarrow & Turnbull, address the valuation and hedging of options using binomial trees. Problems 5 and 6, drawn from Hull, address lognormal price dynamics. Problem 7, adapted from Wilmott-Howison-Dewynne, examines the continuous-time limit more deeply.

We assume throughout that the risk-free rate r is constant and the risky asset has lognormal dynamics with constant drift μ and volatility σ (i.e. $\log s(t_2) - \log s(t_1)$ has mean $\mu(t_2 - t_1)$ and variance $\sigma^2(t_2 - t_1)$).

For your binomial trees in Problems 1-4 please use $s_{\text{up}} = us_{\text{now}}$ and $s_{\text{down}} = ds_{\text{now}}$ with

$$u = e^{[(r-\sigma^2/2)\delta t + \sigma\sqrt{\delta t}]}, \quad d = e^{[(r-\sigma^2/2)\delta t - \sigma\sqrt{\delta t}]}.$$

This is Jarrow and Turnbull's (5.42); the logic behind it is discussed in the Section 4 notes. Convention concerning units: if r and σ are given "per year" then $\delta t = 1/2$ for a time period of 6 months, $\delta t = 1/4$ for a time period of 3 months, etc.

(1) A European put option with strike price 45 dollars matures in one year. The underlying asset has volatility 20 percent per annum, and the current spot price is 50 dollars. The risk-free interest rate is 5.60 percent per annum. Divide the one-year interval into two six-month intervals, and use the recombinant tree with u and d as given above.

- (a) Show that $u = 1.172832$ and $d = .883891$. Evaluate the risk-neutral probabilities.
- (b) Determine the put price by working backward through the tree.
- (c) Determine the put price by using the formula which gives it as an average over all final-time payoffs. Of course your answer should be the same as for (b).
- (d) Describe the associated trading strategy. In other words, specify how many units of stock and how much debt you should hold at each node after rebalancing.

(2) The current stock price is 100, and the volatility is 30 percent per annum. The risk-free interest rate is 6 percent per annum. A one-year European call option is written on this stock with strike price 100. (The writer holds a short call.)

- (a) Divide the one-year period into two six-month intervals, and use the recombinant tree with u and d as given above. Calculate the risk-neutral probabilities.
- (b) The option price is quoted as 14 7/8. Is this consistent with the absence of arbitrage, assuming the recombinant tree of part (a)?
- (c) If your answer to (b) is no, construct an arbitrage that takes advantage of the situation.

(3) A "European powered call" with maturity T and strike price K has payoff $(S_T - K)_+^2$ if S_T where the stock price at maturity. In other words it pays $(S_T - K)^2$ if $S_T > K$ and 0 otherwise. Consider an option of this type with maturity one year and strike price 110. Suppose the stock has volatility 30 percent per annum and the current stock price is 100. The risk-free interest rate is 4.5 percent per annum. Use a two-period recombinant tree, dividing the year into two six-month intervals and defining u, d as above.

(a) Find the risk-neutral probabilities and determine the value of the option. Use the formula giving the value as a weighted sum of payoffs.

(b) Describe the hedge portfolio (the replicating portfolio at time zero).

(c) How does Delta (the hedge ratio, the number of units of stock in the hedge portfolio) compare to that of an ordinary call option? What are the practical implications?

(4) A special kind of one-year put option is written on a stock. The current stock price is 40 and the current strike price is 40. At month 6, if the stock price is below 35 the strike price is lowered to 35; otherwise it remains unchanged. The risk-free interest rate is 5 percent per annum; the volatility of the stock is 35 percent per annum.

(a) Use a 2-period binomial tree with the usual choices of u and d to value the option.

(b) Now use a 4-period binomial tree to value the option.

(c) What is the difficulty with valuing this type of option?

(5) Consider a stock satisfying lognormal price dynamics with drift μ and volatility σ (as defined above). Suppose the stock price now is s_0.

(a) Give a 95% confidence interval for the price at time T, using the fact that with 95% confidence, a Gaussian random variable lies within 1.96 standard deviations of its mean.

(b) Give the mean and variance of the price at time T.

(c) Give a formula for the likelihood that an option with strike price K and maturity T will be in-the-money at maturity.

(d) If the mean return is 16% per annum and the volatility is 30% per annum, what do (a) and (b) tell you about tomorrow's closing price in terms of today's closing price?

(6) Consider a derivative with payoff s_T^n at maturity. Show that its value at time t is

$$s_t^n e^{[\frac{1}{2}\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free rate and σ is the volatility of the underlying asset. (Hint: use the continuous-time pricing formula derived in the Section 4 notes, and the lemma at the end of that section.)

(7) We saw in Section 4 that different binomial trees (associated with different values of μ) can give the same values for options in the continuum limit. So it makes sense to ask: for a given risk-free rate r and volatility σ, which binomial trees give the correct continuum limit? Let us refine this question a bit. We consider only recombinant trees of the form $s_{\text{up}} = us_{\text{now}}$, $s_{\text{down}} = ds_{\text{now}}$. The continuum limit corresponds to $n \to \infty$ time steps of size $\delta t = T/n$. We expect u and d to depend on n, i.e. $u = u_n$, $d = d_n$. For any fixed n the value of the option is $e^{-rT} E_{\text{RN}}[f(s_T)]$; this is the value obtained by working backward through the tree, using the risk-neutral probability $q = q_n = (e^{r\delta t} - d)/(u - d)$. In the continuum limit we know the value should be $e^{-rT} E[f(s_0 e^X)]$ where X is Gaussian with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Our task is to find conditions on u_n and d_n such that

$$E_{\text{RN}}[f(s_T)] \to E[f(s_0 e^X)] \quad \text{as } n \to \infty. \tag{1}$$

The main point of this problem is to show that (1) holds if $u = u_n$ and $d = d_n$ are chosen so that

$$qu + (1-q)d = e^{r\delta t}, \quad qu^2 + (1-q)d^2 = e^{(2r+\sigma^2)\delta t}. \tag{2}$$

Of course the first relation is equivalent to the definition of the risk-neutral probability $q = q_n$, so only the second relation is new. Notice that (2) gives two equations in three unknowns (u, d, q), so there is one remaining degree of freedom.

(a) Define a_n and b_n by $u_n = e^{a_n}$ and $d_n = e^{b_n}$. Show, by arguing as in the Section 4 notes, that (1) holds if

$$n(q_n a_n + (1-q_n)b_n) \to (r - \tfrac{1}{2}\sigma^2)T \quad \text{and} \quad nq_n(1-q_n)(a_n - b_n)^2 \to \sigma^2 T \tag{3}$$

as $n \to \infty$.

(b) Show, by algebraic manipulation, that (2) is equivalent to

$$u = e^{r\delta t}\left(1 + \sqrt{\frac{1-q}{q}(e^{\sigma^2\delta t} - 1)}\right) \quad d = e^{r\delta t}\left(1 - \sqrt{\frac{q}{1-q}(e^{\sigma^2\delta t} - 1)}\right)$$

so that

$$a_n = r\delta t + \log\left[1 + \sqrt{\frac{1-q_n}{q_n}\omega_n}\right] \quad b_n = r\delta t + \log\left[1 - \sqrt{\frac{q_n}{1-q_n}\omega_n}\right]$$

with $\omega_n = e^{\sigma^2\delta t} - 1$.

(c) Use the Taylor expansion of $\log(1+x)$ near $x = 0$ to verify the limits (3) as $n \to \infty$.

(d) How should we choose u_n and d_n, if we want $q_n = 1/2$ *exactly* for each n?