Derivative Securities – Homework 4 – distributed 10/24/00, due 11/7/00.
Corrected version – Problem 1a was wrong before.
Solutions will be distributed 11/14/00

Problems 1-3 address the Black-Scholes pricing formulas and their consequences. In those problems please assume as usual that (a) the risk-free rate r is constant; (b) the price of the underlying stock is described by a lognormal process with constant drift μ and volatility σ; (c) the stock pays no dividends. All options under consideration are European.

Problems 4-7 address the Ito calculus and its applications. In those problems $w(t)$ is a standard Brownian motion with $w(0) = 0$.

(1) Consider a squared call with strike K and maturity T, i.e. an option whose payoff at maturity is $(s_T - K)_+^2$.

(a) Evaluate its hedge ratio (its "Delta") by differentiating under the integral, then evaluating the resulting expression.

(b) Give a formula for the value of the squared call at time 0, analogous to the standard formula $s_0 N(d_1) - Ke^{-rT}N(d_2)$ for an ordinary call.

[Hint: For part (b) use the fact that $(e^x - K)^2 = e^{2x} - 2Ke^x + K^2$. You could of course differentiate your answer to (b) to find Delta, but that's the hard way.]

(2) Consider a "cash-or-nothing" option with strike price K, i.e. an option whose payoff at maturity is

$$f(s_T) = \begin{cases} 1 & \text{if } s_T \geq K \\ 0 & \text{if } s_T < K \end{cases}$$

It can be interpreted as a bet that the stock will be worth at least K at time T.

(a) Give a formula for its value at time t, in terms of the spot price s_t.

(b) Give a formula for its Delta (i.e. its hedge ratio). How does the Delta behave as t gets close to T?

(c) Why is it difficult, in practice, to hedge such an instrument?

[Comment: Such options are rarely found "naked" but they often arise in "structured products" calling for a fixed payment to be made if an asset price is above a certain value on a certain date. In view of (c) it is not entirely clear that the Black-Scholes valuation formula is valid for such an option. What do you think?)

(3) Suppose r is 5 percent per annum and σ is 20 percent per annum. Let's consider standard put and call options with strike price $K = 50$. Do this problem using the Black-Scholes formulas (not a binomial tree).

(a) Suppose the spot price is $s_0 = 50$ and the maturity is one year. Find the value, Delta, and Vega of the put. Same request for the call.

(b) Graph the value of a European call as a function of the spot price s_0, for several maturities. Display all the graphs on a single set of axes, and comment on the trends they reveal.

(c) Same as (b) but for a European put.

(d) Your answer to (c) should show that the value of the put is lower than $(K - s_0)_+$ for $s_0 < s_*$ and higher for $s_0 > s_*$. Estimate the critical value s_* when the maturity T is 2 years.

[Comment: Use whatever means (matlab, mathematica, spreadsheet) is most convenient, but say briefly what you used. One point of this problem is to visualize the behavior of the Black-Scholes pricing formulas. Another is to be sure you have a convenient tool for exploring further on your own.]

(4) We showed in class using Ito's formula that $s(t) = s(0)e^{\mu t + \sigma w(t)}$ solves the stochastic differential equation

$$ds = (\mu + \frac{1}{2}\sigma^2)sdt + \sigma s dw$$

with initial condition $s(0) = s_0$.

(a) Use this to show that $E[s(t)] - E[s(0)] = (\mu + \frac{1}{2}\sigma^2)\int_0^t E[s(\tau)]d\tau$, where E denotes expected value.

(b) Conclude that $E[s(t)] = s(0)e^{(\mu + \frac{1}{2}\sigma^2)t}$.

[Comment: taking $t = 1$, this gives a new proof of the lemma, stated at the end of the Section 4 notes, that if X is Gaussian with mean μ and standard deviation σ then $E[e^X] = e^{\mu + \sigma^2/2}$.]

(5) This problem should help you understand Ito's formula. If w is Brownian motion, then Ito's formula tells us that $z = w^2$ satisfies the stochastic differential equation $dz = 2wdw + dt$. Let's see this directly:

(a) Suppose $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$. Show that $w^2(t_{i+1}) - w^2(t_i) = 2w(t_i)(w(t_{i+1}) - w(t_i)) + (w(t_{i+1}) - w(t_i))^2$, whence

$$w^2(b) - w^2(a) = 2\sum_{i=0}^{N-1} w(t_i)(w(t_{i+1}) - w(t_i)) + \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$$

(b) Let's assume for simplicity that $t_{i+1} - t_i = (b - a)/N$. Find the mean and variance of $S = \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$.

(c) Conclude by taking $N \to \infty$ that

$$w^2(b) - w^2(a) = 2\int_a^b w\, dw + (b - a).$$

[Comment: we did parts of this calculation in the notes and in class, but because it's so enlightening I'm asking you to go through it carefully here.]

(6) Here's another cute application of the Ito calculus. Let

$$\beta_k(t) = E[w^k(t)]$$

where $w(t)$ is Brownian motion (with $w(0) = 0$). Show using Ito's formula that for $k = 2, 3, \ldots$,

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds.$$

Deduce that $E[w^4(t)] = 3t^2$. What is $E[w^6(t)]$?

[Comment: the moments of w can also be calculated from its distribution function, since $w(t)$ is Gaussian with mean 0 and variance t. But the method in this problem is easier, and good practice with Ito's lemma.)

(7) You should have learned in calculus that the deterministic ODE $dy/dt + Ay = f$ can be solved explicitly when A is constant: just multiply by e^{At} to see that $d(e^{At}y)/dt = e^{At}f$ then integrate in time. Let's use a similar trick to solve the stochastic differential equation

$$dy = -cy\,dt + \sigma\,dw, \quad y(0) = y_0,$$

which is known as the Ornstein-Uhlenbeck equation. (Note: this is not the lognormal process, since there is no coefficient of y in the dw term.)

(a) Show using Ito's lemma applied to the function $f(y, z) = yz$ that in general $d(yz) = ydz + zdy + dydz$ but when z is deterministic this reduces to $d(yz) = ydz + zdy$.

(b) Apply this with $z = e^{ct}$ to see that

$$d(e^{ct}y) = ce^{ct}y\,dt + e^{ct}\,dy = e^{ct}\sigma\,dw.$$

(c) Conclude that

$$y(t) = e^{-ct}y_0 + \sigma\int_0^t e^{-c(t-\tau)}\,dw(\tau).$$

(d) What is the mean $E[(y(t)]$?