Derivative Securities – Homework 5 – distributed 11/7/00, due 11/21/00
Solutions will be distributed 11/28/00

1) We considered, in HW3, a derivative whose payoff was (the dollar value of) $s^n(T)$ at maturity, where $s(t)$ has lognormal dynamics and the risk-free rate r is constant. We showed there that the derivative has value

$$s^n(t)e^{[\frac{1}{2}\sigma^2 n(n-1)+r(n-1)](T-t)}.$$

Let's give a different derivation of the same result, using the Black-Scholes PDE.

(a) Substitute $V(s,t) = h(t)s^n$ into the Black-Scholes PDE. What ODE must $h(t)$ solve? What is the appropriate final-time condition?

(b) Verify that $h(t) = e^{[\frac{1}{2}\sigma^2 n(n-1)+r(n-1)](T-t)}$ solves the ODE you found in (a), with the appropriate final-time condition.

(2) Suppose the risk-free rate $r(t)$ and volatility $\sigma(t)$ vary smoothly with t, so the Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(t)s^2\frac{\partial^2 V}{\partial s^2} + r(t)s\frac{\partial V}{\partial s} - r(t)V = 0.$$

Consider a European option with maturity T and payoff $f(s)$. Show that its value at time t can be obtained as follows:

- find the average risk-free rate from t to T: $\bar{r} = \frac{1}{T-t}\int_t^T r(\tau)\,d\tau$;
- find the root mean square volatility from t to T: $\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma^2(\tau)\,d\tau$;
- V equals the value of an option with the same payoff and maturity on an asset with constant volatility $\bar{\sigma}$ in an economy with constant risk-free rate $\bar{r}$. [Hint: one solution is given in section 6.5 of Wilmott et al.]

(3) Let's value a *perpetual American put* with strike K, written on a non-dividend-paying stock with lognormal dynamics. By definition this instrument never matures, and it can be exercised at any time t yielding payoff $(K - s(t))_+$. Since it never matures, we expect its value to be a function of $s(t)$ alone, i.e. to have the form $V(s(t))$ where

$$\tfrac{1}{2}V_{ss}\sigma^2 s^2 + rsV_s - rV \le 0 \quad \text{and} \quad V(s) \ge (K-s)_+$$

for all s. Moreover we expect that there is an exercise threshold s_* such that

$$\tfrac{1}{2}V_{ss}\sigma^2 s^2 + rsV_s - rV = 0 \quad \text{for } s \ge s_*,$$

and

$$V(s) = (K-s)_+ \quad \text{for } s \le s_*,$$

and we expect that V and V_s are continuous at s_*. Finally we expect that $V(s) \to 0$ as $s \to \infty$.

(a) Show that if $k = 2r/\sigma^2$ then $f(s) = As + Bs^{-k}$ solves the PDE $\frac{1}{2}f''\sigma^2 s^2 + rsf' - rf = 0$ for any choice of the constants A and B. (If you know some PDE then you'll recognize that this is the most general possible solution.)

(b) Show that to have $f(s_*) = (K - s_*)$ and $f(\infty) = 0$ we must set $A = 0$ and $B = s_*^k(K - s_*)$.

(c) Show that the choice $s_* = \frac{k}{1+k}K$ gives $f'(s_*) = -1$, and that the resulting function $f(s)$ has all the properties listed above for $V(s)$. This is the desired function $V(s)$ which values the perpetual American put.

(4) [Jarrow-Turnbull, chapter 7, problem 2]. An American put option with a maturity of one year and a strike price of 60 is written on a non-dividend-paying stock. Assume the current stock price s_0 is 60, the volatility σ is 35 percent per year, and the risk-free rate r is 6 percent per year. To keep things simple, let's use a two-period binomial tree to value the option.

(a) Construct an appropriate two-period recombining price tree using $u = \exp[(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}]$ and $d = \exp[(r - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}]$. Notice that when δt is 6 months, $u = 1.2800$, $d = .7803$, and the risk-neutral probability of the up state is $q = .5007$. (You may round this to $q = .5$.) Value the option by working backward through the tree. Don't forget to check whether early exercise is optimal.

(b) Describe the replicating portfolio at each node. Verify that the associated trading strategy is self-financing, and that it replicates the payoff.

(5) [Jarrow-Turnbull, chapter 11, problem 5]. Consider a call option written on French Francs with a maturity of one year. The spot exchange rate is 0.1760 dollars/FF, its volatility is 12 percent per year, and the strike price is 0.16. The contract size is 250,000FF. To keep things simple, let's use a two-period binomial tree to value the option. Assume that if you invest one dollar for six months at the risk-free rate it will be worth 1.0151 dollars, and if you invest one Franc for six months at the risk-free rate it will be worth 1.0305 Francs.

(a) Construct an appropriate two-period recombining price tree using $u = \exp[(r - D - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}] = 1.068436$ and $d = \exp[(r - D - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}] = .901667$.

(b) If the option is European (exercisable only at maturity) what is its value? (You may round off the risk-neutral probability to $q = .5$).

(c) If the option is American (exercisable at six months or at maturity) what is its value?