Derivative Securities – Section 1
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Forwards, puts, calls, and other contingent claims. This section discusses the most basic examples of contingent claims, and explains how considerations of arbitrage determine or restrict their prices. This material is in Chapters 2 and 3 of Jarrow and Turnbull, and Chapters 1, 3 and 7 of Hull. We concentrate for simplicity on European options rather than American ones, on forwards rather than futures, and on deterministic rather than stochastic interest rates.

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The most basic instruments:

Forward contract with maturity T and delivery price K.

buy a forward ↔ hold a long forward
↔ holder is obliged to buy the underlying asset at price K on date T.

European call option with maturity T and strike price K.

buy a call ↔ hold a long call
↔ holder is entitled to buy the underlying asset at price K on date T.

European put option with maturity T and strike price K.

buy a put ↔ hold a long put
↔ holder is entitled to sell the underlying asset at price K on date T.

These are *contingent claims*, i.e. their value at maturity is not known in advance. Payoff formulas and diagrams (value at maturity, as a function of S_T=value of the underlying) are shown in the Figure.

Any long position has a corresponding (opposite) *short* position:

Buyer of a claim has a long position ↔ seller has a short position.

Payoff diagram of short position = negative of payoff diagram of long position.

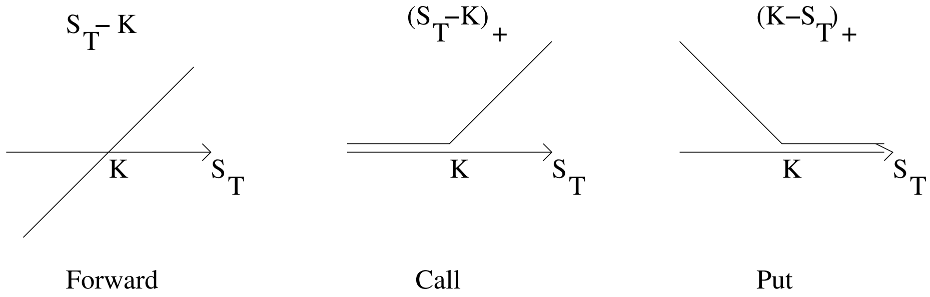

Figure 1: Payoffs of forward, call, and put options.

An *American* option differs from its European sibling by allowing early exercise. For example: the holder of an American call with strike K and maturity T has the right to purchase the underlying for price K at any time $0 \leq t \leq T$. A discussion of American options must deal with two more-or-less independent issues: the unknown future value of the underlying, and the optimal choice of the exercise time. By focusing initially on European options we'll develop an understanding of the first issue before addressing the second.

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Why do people buy and sell contingent claims? Briefly, to *hedge* or to *speculate*. Examples of hedging:

- A US airline has a contract to buy a French airplane for a price fixed in FF, payable one year from now. By going long on a forward contract for FF (payable in dollars) it can eliminate its foreign currency risk.

- The holder of a forward contract has unlimited downside risk. Holding a call limits the downside risk (but buying a call with strike K costs more than buying the forward with delivery price K). Holding one long call and one short call costs less, but gives up some of the upside benefit:

  $$(S_T - K_1)_+ - (S_T - K_2)_+ \qquad K_1 < K_2$$

  This is known as a "bull spread". (See the figure.)

Options are also frequently used as a means for speculation. Basic reason: the option is more sensitive to price changes than the underlying asset itself. Consider for example a European call with strike $K = 50$, at a time t so near maturity that the value of the option is essentially $(S_t - K)_+$. Let $S_t = 60$ now, and consider what happens when S_t increases by 10% to 66. The value of the option increases from about $60 - 50 = 10$

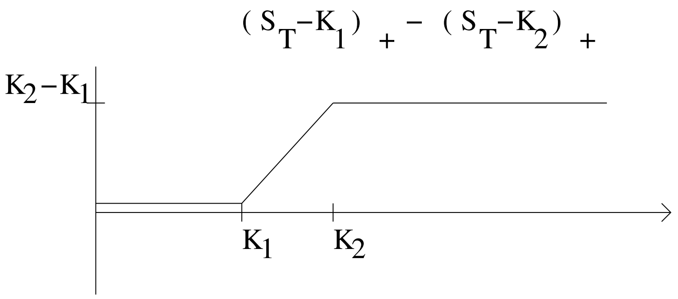

Figure 2: Payoff of a bull spread.

to about $66 - 50 = 16$, an increase of 60%. Similarly if S_t decreases by 10% to 54 the value of the option decreases from 10 to 4, a loss of 60%. This calculation isn't special to a call: almost the same calculation applies to stock bought with borrowed funds. Of course there's a difference: the call has more limited downside exposure.

We assumed the time t was very near maturity so we could use the payoff $(S_T - K)_+$ as a formula for the value of the option. But the idea of the preceding paragraph applies even to options that mature well in the future. We'll study in this course how the Black-Scholes analysis assigns a value $c = c[S_t; T - t, K]$ to the option, as a function of its strike K, its time-to-maturity $T - t$ and the current stock price S_t. The graph of c as a function of S_t is roughly a smoothed-out version of the payoff $(S_t - K)_+$.

Don't be confused: our assertion that "the option is more sensitive to price changes than the underlying asset itself" does *not* mean that $\partial c/\partial S$ is bigger than 1. This expression, which gives the sensitivity of the option to change in the underlying, is called Δ. At maturity the call has value $(S_T - K)_+$ so $\Delta = 1$ for $S_T > K$ and $\Delta = 0$ for $S_T < K$. Prior to maturity the Black-Scholes theory will tell us that Δ varies smoothly from nearly 0 for $S_t \ll K$ to nearly 1 for $S_t \gg K$.

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Some pricing principles:

- If two portfolios have the same payoff then their present values must be the same.
- If portfolio 1's payoff is always at least as good as portfolio 2's, then present value of portfolio 1 $\geq$ present value of portfolio 2.

We'll see presently that these principles must hold, because if they didn't the market would support arbitrage.

First example: value of a forward contract. We assume for simplicity:

(a) underlying asset pays no dividend and has no carrying cost (e.g. a non-dividend-paying stock);

(b) time value of money is computed using compound interest rate r, i.e. a guaranteed income of D dollars time T in the future is worth $e^{-rT}D$ dollars now.

The latter hypothesis amounts to introducing one more investment option:

Bond worth D dollars at maturity T

$$\begin{aligned} \text{buy a bond} &\leftrightarrow \text{hold a long bond} \\ &\leftrightarrow \text{lend } e^{-rT}D \text{ dollars, to be repaid at time T with interest.} \end{aligned}$$

Consider these two portfolios:

Portfolio 1 – one long forward with maturity T and delivery price K, payoff $(S_T - K)$.

Portfolio 2 – long one unit of stock (present value S_0, value at maturity S_T) and short one bond (present value $-Ke^{-rT}$, value at maturity $-K$).

They have the same payoff, so they must have the same present value. Conclusion:

$$\text{Present value of forward} = S_0 - Ke^{-rT}.$$

In practice, forward contracts are normally written so that their present value is 0. This fixes the delivery price, known as the *forward price*:

$$\text{forward price} = S_0 e^{rT} \text{ where } S_0 \text{ is the spot price.}$$

We can see why the "pricing principles" enunciated above must hold. If the market price of a forward were different from the value just computed then there would be an arbitrage opportunity:

$$\begin{aligned} \text{forward is overpriced} &\rightarrow \text{sell portfolio 1, buy portfolio 2} \\ &\rightarrow \text{instant profit at no risk} \\ \text{forward is underpriced} &\rightarrow \text{buy portfolio 1, sell portfolio 2} \\ &\rightarrow \text{instant profit at no risk.} \end{aligned}$$

In either case, market forces (oversupply of sellers or buyers) will lead to price adjustment, restoring the price of a forward to (approximately) its no-arbitrage value.

Second example: put–call parity. Define

$$\begin{aligned} p[S_0, T, K] &= \text{price of European put when spot price is} \\ & \quad\ S_0\text{, strike price is } K\text{, maturity is } T \\ c[S_0, T, K] &= \text{price of European call when spot price is} \\ & \quad\ S_0\text{, strike price is } K\text{, maturity is } T. \end{aligned}$$

The Black-Scholes model gives formulas for p and c based on a certain model of how the underlying security behaves. But we can see now that p and c are related, without knowing anything about how the underlying security behaves (except that it pays no dividends and has no carrying cost). "Put-call parity" is the relation

$$c[S_0, T, K] - p[S_0, T, K] = S_0 - Ke^{-rT}.$$

To see this, compare

Portfolio 1 – one long call and one short put, both with maturity T and strike K; the payoff is $(S_T - K)_+ - (K - S_T)_+ = S_T - K$.

Portfolio 2 – a forward contract with delivery price K and maturity T. Its payoff is also $S_T - K$.

These portfolios have the same payoff, so they must have the same present value. This justifies the formula.

Third example: The prices of European puts and calls satisfy

$$c[S_0, T, K] \geq (S_0 - Ke^{-rT})_+ \quad \text{and} \quad p[S_0, T, K] \geq (Ke^{-rT} - S_0)_+.$$

To see the first relation, observe first that $c[S_0, T, K] \geq 0$ by optionality – holding a long call is never worse than holding nothing. Observe next that $c[S_0, T, K] \geq S_0 - Ke^{-rT}$, since holding a long call is never worse than holding the corresponding forward contract. Thus $c[S_0, T, K] \geq \max\{0, S_0 - Ke^{-rT}\}$, which is the desired conclusion. The argument for the second relation is similar.

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Note some hypotheses underlying our discussion:

- no transaction costs; no bid-ask spread;
- no tax considerations;
- unlimited possibility of long and short positions; no restriction on borrowing.

These are of course merely approximations to the truth (like any mathematical model). More accurate for large institutions than for individuals.

Note also some features of our discussion: We are simply reaping consequences of the hypothesis of no arbitrage. Conclusions reached this way don't depend at all on what you think the market will do in the future. Arbitrage methods restrict the prices of (related) instruments. On the other hand they don't tell an individual investor how best to invest his money. That's the issue of portfolio optimization, which requires an entirely different type of analysis and is discussed in the course Capital Markets and Portfolio Theory.

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A word about interest rates. In the real world interest rates change unpredictably. And the rate depends on maturity. In discussing forwards and European options this isn't particularly important: all that matters is the cost "now" of a bond worth one dollar at maturity T. Up to now we wrote this as e^{-rT}. When multiple borrowing times and maturities are being considered, however, it's clearer to use the notation

$$B(t,T) = \text{cost at time } t \text{ of a risk-free bond worth 1 dollar at time } T.$$

In a constant interest rate setting $B(t,T) = e^{-r(T-t)}$. If the interest rate is nonconstant but deterministic – i.e. known in advance – then an arbitrage argument shows that $B(t_1,t_2)B(t_2,t_3) = B(t_1,t_3)$. If however interest rates are stochastic – i.e. if $B(t_2,t_3)$ is not known at time t_1 – then this relation must fail, since $B(t_1,t_2)$ and $B(t_1,t_3)$ are (by definition) known at time t_1.

Since our results on forwards, put-call parity, etc. used only one-period borrowing, they remain valid when the interest rate is nonconstant and even stochastic. For example, the value at time 0 of a forward contract with delivery price K is $S_0 - KB(0,T)$ where S_0 is the spot price.

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Forwards versus futures. A future is a lot like a forward contract – its writer must sell the underlying asset to its holder at a specified maturity date. However there are some important differences:

- Futures are standardized and traded, whereas forwards are not. Thus a futures contract (with specified underlying asset and maturity) has a well-defined "future price" that is set by the marketplace. At maturity the future price is necessarily the same as the spot price.

- Futures are "marked to market," whereas in a forward contract no money changes hands till maturity. Thus the value of a future contract, like that of a forward contract, varies with changes in the market value of the underlying. However with a future the holder and writer settle up daily while with a forward the holder and writer don't settle up till maturity.

The essential difference between futures and forwards involves the timing of payments between holder and writer: daily (for futures) versus lump sum at maturity (for forwards). Therefore the difference between forwards and futures has a lot to do with the time value of money. If interest rates are constant – or even nonconstant but deterministic – then an arbitrage-based argument shows that the forward and future prices must be equal. (I like the presentation in Appendix 3B of Hull. It is presented in the context of a constant interest rate, but the argument can easily be modified to handle a deterministically-changing interest rate.)

If interest rates are stochastic, the arbitrage-based relation between forwards and futures breaks down, and forward prices can be different from future prices. In practice they are different, but usually not much so.

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A word about taxes. Tax considerations are not always negligible. Here are two examples, each closely related to put-call parity.

Constructive sales. An investor holds stock in XYZ Corp. His stock has appreciated a lot, and he thinks it's time to sell, but he wishes to postpone his gain till next year when he expects to have losses to offset them. Prior to 1997 he could have (1) kept his stock, (2) bought a put (one year maturity, strike K), (3) sold a call (one year maturity, strike K), and (4) borrowed Ke^{-rT}. The value of this portfolio at maturity is $S_T + (K - S_T)_+ - (S_T - K)_+ - K = 0$. Since his position at time T is valueless and risk-free, he would have effectively "sold" his stock. Since the present value of items (1)-(4) together is 0, the combined value of the long put, short call, and loan must be the present value of the stock. Thus the investor would have effectively sold the stock for its present market value, while postponing realization of the capital gain till the options matured.

The tax law was changed in 1997 to treat such a transaction as a "constructive sale," eliminating its attractiveness (the capital gain is no longer postponed). A related strategy is still available however: by combining puts and calls with different maturities, an investor can take a position that still has some risk (thus avoiding the constructive sale rule) while locking in most of the gain and avoiding any capital gains tax till the options mature.

Dick Cheney's Halliburton options. Dick Cheney holds "executive stock options" from Halliburton corporation that mature well after the next Presidential inauguration. These are essentially call options, except (a) they can only be exercised, not sold; (b) if they expire in the money, the gain associated with their exercise is taxed as ordinary income, not as a capital gain. An article by Floyd Norris in the Business section of the 8/26/00 New York Times has a good discussion of the details (a copy is on reserve in the "green box" associated with my name). For simplicity let's concentrate on just part of Cheney's portfolio: an option to buy 100,000 shares at K=39.50/share in December 2002. Let's further take the current stock price to be $S_0 = 53.00$/share (that's about right) and let's ignore the time value of money (a minor detail).

Cheney's problem is this: if he gets elected and continues to hold the options to maturity then he'll have a conflict of interest. But he cannot sell the options prior to maturity. And simply disowning them means disowning an asset that's potentially very valuable.

A common – but flawed – suggestion is that Cheney enter into a forward contract to sell his shares at the time of maturity. Since we're ignoring the time-value of money, the forward price is the present market value, i.e. a contract to sell his stock at $S_0 = 53.00$/share in December 2002 has present value 0. This proposal however has two flaws:

(a) it does not fully eliminate his conflict of interest; and

(b) when taxes are taken into account it doesn't even come close to eliminating his conflict of interest.

Concerning (a): the payoff at maturity of the call and forward together is

$$(S_T - K)_+ - (S_T - S_0) = \begin{cases} S_0 - S_T & \text{if } S_T < K \\ S_0 - K & \text{if } S_T > K \end{cases}$$

Thus Cheney's position would not be insensitive to S_T – at least not if $S_T < K$. The best possible result (for him) would be for Halliburton to go bankrupt ($S_T = 0$).

Concerning (b): the after-tax value of the combined call and forward position depends on S_T and also the details of Cheney's finances. The reason is that his profit on the option $(S_T - K)_+$ is treated as ordinary income but his gain or loss on the forward contract is treated as a capital gain or loss. Suppose the stock price goes up, i.e. $S_T > S_0$. Then in 2002 Cheney has ordinary income $S_T - K$ and a capital loss of $S_T - S_0$. US tax law treats the two very differently: individuals are permitted to offset at most \$3000 of ordinary income by capital losses. Thus if Cheney has no capital gains on other investments to offset his capital losses on the forward contract then he'll be taxed on essentially the full gain $S_T - K$ (though he can use the capital loss to offset capital gains in a future year).

Fixing (a) would be easy, if it were not for the constructive-sale rule: the problem is that a call is not equivalent to a foward. Rather, by put-call parity, it is equivalent to a forward plus a put. Of course Cheney could get around the constructive-sale rule by taking a position whose value still has some dependence on S_T (e.g. by using a put with a strike price different from K) – but then he wouldn't have fixed his conflict of interest problem! Fixing (b) seems nearly impossible, since it depends on details of Cheney's financial position that have nothing to do with this transaction.