Derivative Securities – Section 10
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Interest-based instruments: bonds, forward rate agreements, and swaps. This section provides a fast introduction to the basic language of interest-based instruments, then introduces some specific, practically-important examples, including forward rate agreements and swaps. This material can be found in both Hull (chapters 4 and 5) and Jarrow-Turnbull (chapters 13 and 14); personally I find Hull easier to read.

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Bond prices and term structure. The time-value of money is expressed by the *discount factor*

$$B(t, T) = \text{value at time } t \text{ of a dollar received at time } T.$$

This is, by its very definition, the price at time t of a zero-coupon bond which pays one dollar at time T. If interest rates are stochastic then $B(t, T)$ will not be known until time t. Prior to time t it is random – just as in our discussion of equity prices, $s(t)$ was random. Note however that $B(t, T)$ is a function of *two* variables, the initiation time t and the maturity time T. Its dependence on T reflects the *term structure* of interest rates. We usually take the convention that the present time is $t = 0$; thus what is observable now is $B(0, T)$ for all $T > 0$.

There are several equivalent ways to represent the time-value of money. The *yield* $Y(t, T)$ is defined by

$$B(t, T) = e^{-Y(t,T)(T-t)};$$

it is the unique constant interest rate that would have the same effect as $B(t, T)$ under continuous compounding. The *term rate* $R(t, T)$ is defined by

$$B(t, T) = \frac{1}{1 + R(t, T)(T - t)};$$

it is the unique interest rate that would have the same effect as $B(t, T)$ with no compounding. The *instantaneous forward rate* $f(t, T)$ is defined by

$$B(t, T) = e^{-\int_t^T f(t,\tau)\, d\tau};$$

it is unique deterministic time-varying interest rate that describes all the discount factors with initial time t and various maturities.[1] We can easily solve for $Y(t, T)$, $R(t, T)$, or $f(t, T)$ in terms of $B(t, T)$. Therefore each contains the same information as $B(t, T)$ as t and T vary. (Let us also mention one more: the discount rate $I(t, T)$, defined by $B(t, T) = 1 - I(t, T)(T - t)$. It has little conceptual importance; however interest rates for US Treasury bills are usually presented by tabulating these discount rates.)

Most long-term bonds have *coupon payments* as well as a *final payment*. The value of the bond at time 0 is the sum of the present values of all future payments. For a *fixed-rate* bond

[1] Do not confuse this with the forward term rate, introduced below and called $f_0(t, T)$.

the coupon payments (amount c_j at time t_j) are fixed in advance, as is the final payment (amount F at time T). The value of the bond at time t is thus

$$\text{cash price} = \sum c_j B(t, t_j) + F B(t, T).$$

This is known as the *cash price*; it is a consequence of the principle of no arbitrage. Notice that the cash price is a discontinuous function of time: it rises gradually between coupon payments, then falls abruptly at each coupon date t_j because the holder of the bond collects the coupon payment. The cash price is not the value you'll see quoted in the newspaper. What you find there is the difference between the cash price and the interest accrued since the last coupon date:

$$\text{quoted price} = \sum c_j B(t, t_j) + F B(t, T) - \text{accrued interest}.$$

Notice that the quoted price is a continuous function of time, since the accrued interest is discontinuous (it resets to zero at each coupon payment) and the two discontinuities cancel.

A *floating-rate* bond is one whose interest rate (coupon rate) is reset at each coupon date. By definition, after each coupon payment its value returns to its *face value*. A typical example is a one-year floating-rate note with semiannual payments and face value one dollar, pegged to the LIBOR (London Interbank Offer) rate. Suppose at date 0 the LIBOR term interest rate for six-month-maturity is 5.25 percent per annum, but at the six-month reset the LIBOR six-month-maturity rate has changed to 5.6 percent per annum. Then the coupon payment due at six months is $.0525/2 = .02625$, and the coupon payment due at one year is $.056/2 = .028$; in addition the face value (one dollar) is repaid since the bond matures. Note the convention: interest is paid at the end of each period, using the interest rate set at the beginning of the period.

The value of the fixed-rate bond was the discounted value of its future income stream. The same is true of the floating-rate bond, provided that we *discount using the LIBOR rate*. In other words for this purpose $B(t, T)$ should be the value at time t of a LIBOR contract worth one dollar at time T. In fact, the value of the floating-rate bond at six months (just after the first coupon payment) is the value at that time of the payments to be made at one year. If t is six months and T is 1 year then this is

$$B(t, T)(.028 + 1) = \frac{1}{1 + .028}(.028 + 1) = 1.$$

The bond could be sold for this value – so holding it at six months is exactly the same as having one dollar of income at six months. The value of the bond at time 0 is similarly

$$B(0, t)(\text{first coupon} + \text{value at six months}) = \frac{1}{1 + .02625}(.02625 + 1) = 1.$$

Our calculation is clearly not special to the example; it resides in the fact that $B(t, T) = 1/(1 + R(t, T)(T - t))$.

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Forward rates and forward rate agreements. When interest rates are deterministic $B(0,t)B(t,T) = B(0,T)$ (this was on Homework 1). When they are random this is clearly not the case, since $B(0,t)$ and $B(0,T)$ are known at time 0 while $B(t,T)$ is not. However the ratio

$$F_0(t,T) = B(0,T)/B(0,t)$$

still has an important interpretation: it is the discount factor (for time-t borrowing, with maturity T) that can be locked in now, at no cost, by a combination of market positions. In fact, consider the following portfolio:

(a) long a zero-coupon bond worth one dollar at time T (present value $B(0,T)$), and

(b) short a zero-coupon bond worth $B(0,T)/B(0,t)$ at time t (present value $-B(0,T)$).

Its present value is 0, and its holder pays $B(0,T)/B(0,t)$ at time t and receives one dollar at time T. Thus the holder of this portfolio has "locked in" $F_0(t,T)$ as his discount rate for borrowing from time t to time T.

This discussion makes reference to just three times: 0, t and T. So it is natural and conventional to work with term rates rather discount rates. Defining $f_0(t,T)$ by

$$F_0(t,T) = \frac{1}{1 + f_0(t,T)(T-t)}$$

we have shown that $f_0(t,T)$ is the *forward term rate* for borrowing from time t to time T.[2] In other words, an agreement now to borrow or lend later (at time t, with maturity T) has present value zero, if it stipulates that the term rate is $f_0(t,T)$.

What about a contract to borrow or lend at a rate R_K other than $f_0(t,T)$? This is known as a *forward rate agreement.* We can value it by an easy modification of the argument used above. Suppose the principal (the amount to be borrowed at time t) is L. Then the contract provides a payment at time T of

$$(1 + R_K \Delta T)L = (1 + f_0 \Delta T)L + (R_K - f_0)\Delta T L$$

where $f_0 = f_0(t,T)$ and $\Delta T = T - t$. So the contract is equivalent to a forward rate agreement at rate $f_0(t,T)$ on principal L *plus* an additional payment of $(R_K - f_0)\Delta T \cdot L$ at time T. The forward agreement at rate f_0 has present value 0, so the contract's present value (to the lender) is

$$B(0,T)(R_K - f_0)\Delta T \cdot L.$$

The following observations are useful in connection with swaps (which we'll discuss shortly):

(1) *A forward rate agreement is equivalent to an agreement that the lending party may pay interest at the market rate $R(t,T)$ but receive interest at the contract rate R_K.* Indeed, the lender pays L at time t and receives $(1 + R_K\Delta T)L$ at time T. We may suppose that the payment at time t is borrowed at the market rate. Then the lender

[2] Do not confuse this with the instantaneous forward rate discussed earlier.

is (a) borrowing L at the market rate R at time t, repaying $(1 + R\Delta T)L$ at time T, and (b) lending L to the counterparty at time t, receiving repayment $(1 + R_K \Delta T)L$ at time T. Briefly: the lender is *exchanging* the market rate R for the contract rate R_K.

(2) *A forward rate agreement can be priced by assuming that the market rate $R(t, T)$ will be the forward rate $f_0(t, T)$.* Indeed, the pair of loans just considered have net cash flow 0 at time t, and the lender receives $(R_K - R)\Delta T \cdot L$ at time T. The value of R is not known at time 0. But substitution of f_0 in place of R gives the correct value of the contract at time 0.

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Swaps. A *swap* is an exchange of one income or payment stream for another. The most basic example is a (plain vanilla) interest rate swap, which exchanges the cash flow of a floating-rate debt for that of a fixed-rate debt with the same principal. We shall restrict our attention to this case.

A swap is, in a sense, the floating-rate bond analogue of a forward contract. It permits the holder of a floating-rate bond to eliminate his interest-rate risk. This risk arises because the future interest payments on a floating-rate bond are unknown. It can be eliminated by entering into a swap contract, exchanging the income stream of the floating-rate bond for that of a fixed-rate bond. What fixed rate to use? Any rate is possible – but in general the associated swap contract will have some (positive or negative) value. However at any given time there is a fixed rate that sets the present value of the swap to 0. This is rate that would normally be used. Jarrow and Turnbull call it the *par swap rate*.

It is clear from the definition that a swap is equivalent to a portfolio of two bonds, one short and the other long, one a fixed-rate bond and the other a floating-rate bond. Real bonds would have coupon payments then would return the principal at maturity. In a swap the coupon payments don't match, so there is a cash flow at each coupon date; however the principals do match, so there is no net cash flow at maturity. But the principal of the associated bonds isn't irrelevant – we need it to calculate the interest payments. It is called the *notional principal* of the swap.

A swap can also be viewed as a collection of forward rate agreements. Indeed, we showed above that the value of a floating-rate bond is equal to its principal just after each reset. So being short the floating-rate bond and long the fixed-rate bond is equivalent to paying the market interest rate and receiving the fixed interest rate. This amounts to a collection of forward rate agreements – one for each coupon payment – all with the same principal (the notional principal of the swap) and the same interest rate (the fixed rate of the swap).

Valuing a swap is easy: it suffices to value each associated bond then take the difference. (An alternative, equivalent procedure is to value each associated forward rate agreement and add them up.) The following example is a slightly modified version of the one in Jarrow-Turnbull Section 14.1. Suppose an institution receives fixed payments at 7.15% per annum and floating payments determined by LIBOR. We assume there are two payments per year, the maturity is two years, and the notional principal is N. To value the fixed side of the swap we must find the present value of the future coupon payments. It is natural to use the LIBOR discount rate for $B(0,T)$ (Hull makes this choice) though it would be possible to use the treasure-bill discount rates instead (Jarrow-Turnbull makes this choice). Let us assume

$$B(0,t_1) = .9679, \quad B(0,t_2) = .9362, \quad B(0,t_3) = .9052, \quad B(0,t_4) = .8749$$

where $t_1 = 182$ days, $t_2 = 365$ days, $t_3 = 548$ days, and $t_4 = 730$ days are the precise payment dates. The value of the fixed side of the swap is then

$$\begin{aligned} V_{\text{fix}} &= N\{.9679 \times .0715 \times (182/365) + .9362 \times .0715 \times (183/365) \\ &\quad + .9052 \times .0715 \times (183/365) + .8749 \times .0715 \times (182/365)\} = (0.1317)N \end{aligned}$$

Notice that we counted only the coupon payments, with no final payment of principal.

Now let's value the floating side of the swap. Of course we cannot know its cash flows at each time – this would require knowledge of $B(t_i, t_{i+1})$ for each i, which cannot be known at time 0. However to value the swap all we really need to know is $B(0,t_4)$. Indeed, the value of the floating bond at time 0 is just its notional principal N. But we did not count the return of principal V_{fix}, so we must not count it here either. Thus the value of the floating side of the swap is

$$V_{\text{float}} = N - B(0,t_4)N = (0.1251)N.$$

The value of the swap is the difference, namely

$$V_{\text{swap}} = V_{\text{fix}} - V_{\text{float}} = (0.0066)N.$$

This is, of course, the value of the swap to the party receiving the fixed rate and paying the floating rate. The value to the other party is $-(0.0066)N$.

OK, that was easy. But the answer didn't come out zero. What fixed rate could have been used to make the answer come out zero – in other words, what is the par swap rate? That's easy: we must replace .0715 in the above by a variable x, set the value of the swap to 0, and solve for x. This gives

$$x \cdot \{.9679 \times (182/365) + .9362 \times (183/365) + .9052 \times (183/365) + .8749 \times (182/365)\} = 0.1251,$$

which simplifies to $1.8421x = 0.1251$ whence $x = .0679$. Thus the par swap rate is 6.79% per annum.

We have discussed only the simplest kind of swap – a "plain vanilla interest rate swap". But the general principle should be clear. Another widely used instrument is the "plain vanilla foreign currency swap," which exchanges a fixed-rate income stream in a foreign currency for a fixed-rate income stream in dollars. Such an instrument can be used to eliminate foreign currency risk. See Jarrow-Turnbull section 14.2 for a discussion of its valuation.

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Forwards versus futures. There is a well-developed market for futures contracts on treasury bonds. At first this may seem surprising, since there are so many different types of bonds and a futures contract must refer to a well-defined underlying. In practice this difficulty is avoided by rules that permit a variety of similar bonds to be delivered when the contract matures, with cash adjustments depending on the specific bond delivered. This feature makes the futures market complicated and interesting.

Here however we wish to focus on a different issue, namely the relationship between forward and future prices. (This discussion follows section 12.3 of the book by Avellaneda and Laurence. Note however that what they call $F(t,T)$ is what we are calling $f_0(t,T)$.) Our purpose is partly to emphasize that the two are different, and partly to get a handle on how the dynamics of interest rates determines forward rates.

We have in mind the binomial-tree, risk-neutral-expectation setting explained at the end of Section 9. However we shall use the notation of a continuous-time model (mainly: integrals rather than sums) since this is less cumbersome. Our starting point is the fact that

$$B(t,T)/A(t) = E_{\mathrm{RN}}\left[1/A(T)\right]$$

where $A(t)$ is the value of the money-market fund at time t. In the continuous time setting $A(T) = A(t)\exp\int_t^T r(s)\,ds$ so the preceding formula becomes

$$B(t,T) = E_{\mathrm{RN}}\left[e^{-\int_t^T r(s)\,ds}\right].$$

A typical interest rate future involves 3-month Eurodollar contracts: at the contract's maturity the holder must make a 3-month loan to the counterparty, at interest rate equal to the 3-month-term LIBOR rate. We have called this rate $R(t,T)$, where $T{=}t+3$ months, and t is the maturity date of the futures contract. We know from Section 9 that the associated futures price $\tilde{f}_0(t,T)$ – which determines the daily settlements during the course of the contract – is a martingale, in other words

$$\tilde{f}_0(t,T) = E_{\mathrm{RN}}\left[R(t,T)\right].$$

Let us seek a similar representation for the forward term rate $f_0(t,T)$, defined as above by

$$\frac{1}{1+f_0(t,T)\Delta T} = F_0(t,T) = \frac{B(0,T)}{B(0,t)}$$

with $\Delta T = T-t$. Solving for $f_0(t,T)$ gives

$$f_0(t,T) = \frac{1}{\Delta T\cdot B(0,T)}(B(0,t)-B(0,T)).$$

Rewriting the expression in parentheses as a risk-neutral expectation gives

$$\begin{aligned}
f_0(t,T) &= \frac{1}{\Delta T\cdot B(0,T)}E_{\mathrm{RN}}\left[e^{-\int_0^t r(s)\,ds} - e^{-\int_0^T r(s)\,ds}\right]\\
&= \frac{1}{B(0,T)}E_{\mathrm{RN}}\left[e^{-\int_0^t r(s)\,ds}\cdot\frac{1-e^{-\int_t^T r(s)\,ds}}{\Delta T}\right]\\
&= \frac{1}{B(0,T)}E_{\mathrm{RN}}\left[e^{-\int_0^t r(s)\,ds}\cdot\frac{1-B(t,T)}{\Delta T}\right],
\end{aligned}$$

making use in the last step of the fact that risk-neutral expectations are determined working backward in time. Now, the relation $B(t,T) = 1/[1 + R(t,T)\Delta T]$ can be rewritten as

$$\frac{1 - B(t,T)}{\Delta T} = R(t,T)B(t,T),$$

so we have shown that

$$\begin{aligned} f_0(t,T) &= \frac{1}{B(0,T)} E_{\mathrm{RN}} \left[e^{-\int_0^t r(s)\,ds} R(t,T)B(t,T) \right] \\ &= \frac{1}{B(0,T)} E_{\mathrm{RN}} \left[e^{-\int_0^t r(s)\,ds} R(t,T) e^{-\int_t^T r(s)\,ds} \right], \end{aligned}$$

using once more the fact that risk-neutral expectations are determined working backward in time. Combining the two exponential terms, we conclude finally that

$$f_0(t,T) = \frac{E_{\mathrm{RN}} \left[R(t,T) e^{-\int_0^T r(s)\,ds} \right]}{E_{\mathrm{RN}} \left[e^{-\int_0^T r(s)\,ds} \right]}.$$

Thus the forward rate $f_0(t,T)$ is *not* the risk-neutral expectation of the term rate $R(t,T)$. Rather it is the expectation of $R(t,T)$ with respect to a different probability measure, the one obtained by weighting each path by $\exp\left(-\int_0^T r(s)\,ds\right)$.

It is clear from this calculation that forward rates and futures prices are different. We can also see something about the relation between the two. In fact, writing $R = R(t,T)$ and $D = \exp\left(-\int_0^T r(s)\,ds\right)$ we have

$$\text{forward rate} - \text{futures price} = \frac{E[RD] - E[R]E[D]}{E[D]}.$$

where E represents risk-neutral expectation. If R and D were independent the right hand side would be zero and forward rates would equal futures prices. In general however we should expect R and D to be negatively correlated, since R is a term interest rate and D is a discount factor. Recognizing that $E[RD] - E[R]E[D]$ is the covariance of R and D, we conclude that this expression should normally be negative, implying that

$$\text{forward rate} < \text{futures price}.$$

This is in fact what is observed (the difference is relatively small). A scheme for adjusting the futures price to obtain the forward rate is sometimes called a "convexity adjustment". It should be clear from our analysis that different models of stochastic interest rate dynamics lead to different convexity adjustment rules.

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Caps, floors, and swaptions. Easiest first: a swaption is just an option on a swap. When it matures, its holder has the right to enter into a specified swap contract. He'll do so of course only if this swap contract has positive value. Since a swap is equivalent to a pair of bonds, a swaption can be viewed as an option on a pair of bonds. Similarly, since a swap is equivalent to a collection of forward rate agreements, a swaption can be viewed as an option on a collection of forward rate agreements.

Now let's discuss caps. The borrower in a floating-rate loan does not know his future expenses, since they depend on the floating interest rate. He could eliminate this uncertainty entirely by entering into a swap agreement. But suppose all he wants is insurance against the worst-case scenario of a high interest rate. The cap was invented for him: it pays the difference between the market interest rate and a specified cap rate at each coupon date, if this difference is positive. By purchasing a cap, the borrower insures in effect that he'll never have to pay an interest rate aboved the cap rate. The cap can be viewed as a collection of caplets, one associated with each coupon payment. Each caplet amounts to an option on a bond. It is roughly speaking a call option on the market rate at the coupon time.

A floor is like a cap, but it insures a sufficiently high interest rate rather than a sufficiently low one. It can be viewed as a collection of floorlets, one associated with each coupon payment. Each floorlet is again an option on a bond – roughly speaking a put option on the market rate at the coupon time.

There is a version of put-call parity in this setting: cap-floor=swap, if the fixed rate specified by all three instruments is the same.

Thus caps and floors are collections of options on bonds; swaptions are options on collections of bonds. We'll discuss them in more detail in the next section, and we'll explain how they can be priced using a variant of Black's formula.