Derivative Securities – Section 3
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Multiperiod Binomial Trees. We turn to the valuation of derivative securities in a time-dependent setting. We focus for now on multi-period binomial models, i.e. binomial trees. This setting is simple enough to let us do everything explicitly, yet rich enough to approximate many realistic problems.

The material covered in this section is very standard (and very important). The treatment here is essentially that of Baxter and Rennie (Chapter 2). The same material is also in Jarrow and Turnbull (Chapter 5) and Hull (Chapter 9). Jarrow and Turnbull has the advantage of including many examples. In the next section of notes we'll discuss how the parameters should be chosen to mimic the conventional (Black-Scholes) hypothesis of lognormal stock prices, and we'll pass to the continuous-time limit.

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The multi-period binomial model generalizes the single-period binomial model we considered in Section 2. It has

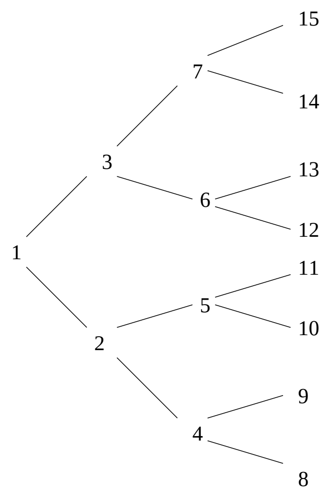

Figure 1: States of a non-recombining binomial tree.

- just two securities: a risky asset (a "stock," paying no dividend) and a riskless asset ("bond");
- a series of times $0, \delta t, 2\delta t, \ldots, N\delta t = T$ at which trades can occur;
- interest rate r_i during time interval i for the bond;
- a binomial tree of possible states for the stock prices.

The last statement means that for each stock price at time $j\delta t$, there are two possible values it can take at time $(j+1)\delta t$ (see Figure 1).

The interest rate environment is described by specifying the interest rates r_i. We restrict our attention for now to the case of a constant interest rate: $r_i = r$ for all i.

The stock price dynamics is described by assigning a price s_j to each state in the tree. Strictly speaking we should also assign (subjective) probabilities p_j to the branches (the two branches emerging from a given node should have probabilities summing to 1): see Figure 2.

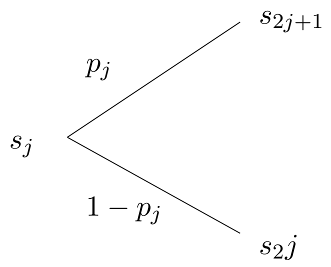

Figure 2: One branch of a binomial tree.

Actually, we will make no use of the subjective probabilities p_j; our arguments are based on arbitrage, so they depend only on the list of possible states not on their probabilities. However our pricing formula will make use of *risk-neutral probabilities* q_j. These "look like" subjective probabilities, except that they are determined by the stock prices and the interest rate.

Our stock prices must be "reasonable" in the sense that the market support no arbitrage. Motivated by the one-period model, we (correctly) expect this condition to take the form:

- starting from any node, the stock price may do better than or worse than the risk-free rate during the next period.

In other words, $s_{2j} < e^{r\delta t} s_j < s_{2j+1}$ for each j.

The tree in Figure 1 is the most general possible. At the nth time step it has 2^n possible states. That's a lot of states, especially when n is large. It's often convenient to let selected states have the same prices in such a way that the list of distinct prices forms a *recombinant tree*. Figure 3 gives an example of a 4-stage recombinant tree, with stock prices marked for each state: (A recombinant tree has just $n+1$ possible states at time step n.)

A special class of recombinant trees is obtained by assuming the stock price goes up or down by fixed multipliers u or d at each stage: see Figure 4. This last class may seem terribly special relative to the general binomial tree. But we shall see it is general enough for many practical purposes – just as a random walk (consisting of many steps, each of fixed magnitude but different in direction) can approximate Brownian motion. And it has the advantage of being easy to specify – one has only to give the values of u and d.

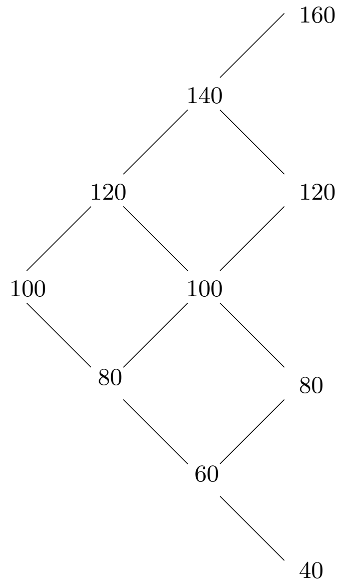

Figure 3: A simple recombinant binomial tree.

It may seem odd that we consider a market with just one stock, when real markets have many stocks. But our goal is to price contingent claims based on considerations of arbitrage. If we succeed using just these two instruments (the stock and the riskless bond) then our conclusions necessarily apply to any larger market containing both instruments.

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Our goal is to determine the value (at time 0) of a contingent claim. We will consider American options later; for the moment we consider only European ones, i.e. early redemption is prohibited. The most basic examples are European calls and puts (payoffs: $(S_T - K)_+$ and $(K - S_T)_+$ respectively). However our method is much more general. What really matters is that *the payoff of the claim depends entirely on the state of stock process at time* T.

Let's review what we found in the one-period binomial model. Our multiperiod model consists of many one-period models, so it is convenient to introduce a flexible labeling scheme. Writing "now" for what used to be the initial state, and "up, down" for what used to be the two final states, our risk-neutral valuation formula was

$$f_{\mathrm{now}} = e^{-r\delta t}[q f_{\mathrm{up}} + (1-q) f_{\mathrm{down}}]$$

where

$$q = \frac{e^{r\delta t} s_{\mathrm{now}} - s_{\mathrm{down}}}{s_{\mathrm{up}} - s_{\mathrm{down}}}.$$

Here we're writing f_{now} for what we used to call $V(f)$, the (present) value of the contingent claim worth f_{up} or f_{down} at the next time step if the stock price goes up or down respectively.

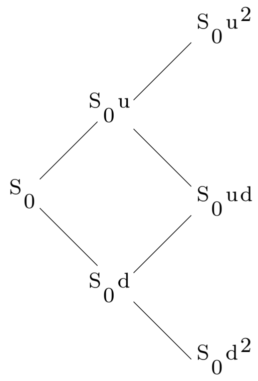

Figure 4: A multiplicative recombinant binomial tree.

This formula was obtained by replicating the payoff with a combination of stock and bond; the replicating portfolio used

$$\phi = \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$$

units of stock, and a bond worth $f_{\text{now}} - \phi s_{\text{now}}$.

The valuation of a contingent claim in the multiperiod setting is an easy consequence of this formula. We need only "work backward through the tree," applying the formula again and again.

Consider, for example, the four-period recombinant tree shown in Figure 3. (This example, taken straight from Baxter and Rennie, has the nice feature of very simple arithmetic.) Suppose the interest rate is $r = 0$, for simplicity. Then $q = 1/2$ at each node (we chose the prices to keep this calculation simple). Let's find the value of a European call with strike price 100 and maturity $T = 3\delta t$. Working backward through the tree:

- The values at maturity are $(S_T - 100)_+ = 60, 20, 0, 0$ respectively.
- The values one time step earlier are 40, 10, and 0 respectively, each value being obtained by an application of the one-period formula.
- The values one time step earlier are 25 and 5.
- The value at the initial time is 15.

Easy. But is it right? Yes, because these values can be replicated. However the replication strategies are more complicated than in the one-period case: the replicating portfolio must be adjusted at each trading time, taking into account the new stock price.

Let's show this in the example. Using our one-period rule, the replicating portfolio starts with $\phi = (25 - 5)/(120 - 80) = .5$ units of stock, worth $.5 \times 100 = 50$ dollars, and a bond worth $15 - 50 = -35$. (Viewed differently: the investment bank that sells the option collects 15 dollars; it should borrow another 35 dollars, and use these 15+35=50 dollars to buy 1/2

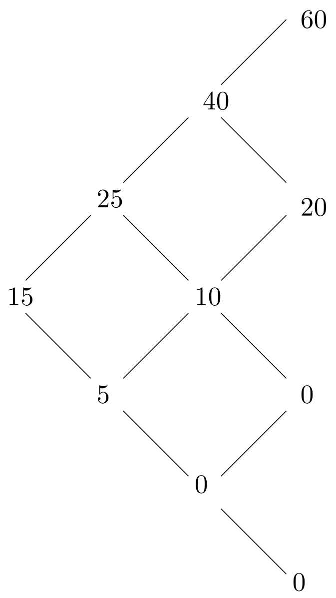

Figure 5: Value of the option as a function of stock price state.

unit of stock.) The claim is that by trading intelligently at each time-step we can adjust this portfolio so it replicates the payoff of the option no matter what the stock price does. Here is an example of a possible history, and how we would handle it:

Stock goes up to 120 The new ϕ is $(40 - 10)/(140 - 100) = .75$, so we need another .25 units of stock. We must buy this at the present price, 120 dollars per unit, and we do it by borrowing 30 dollars. Thus our debt becomes 65 dollars.

Stock goes up again to 140 The new ϕ is $(60 - 20)/(160 - 120) = 1$, so we buy another .25 unit at 140 dollars per unit. This costs another 35 dollars, bringing our debt to 100 dollars.

Stock goes down to 120 At maturity we hold one share of stock and a debt of 100. So our portfolio is worth 120-100 = 20, replicating the option. (Put differently: if the investment bank that sold the option followed our instructions, it could deliver the unit of stock, collect the 100, pay off its loan, and have neither a loss nor a gain.)

That wasn't a miracle. We'll explain it more formally pretty soon. But here's a second example – a different possible history – to convince you:

Stock goes down to 80 The revised ϕ is $(10 - 0)/(100 - 60) = .25$. So we should sell 1/4 unit stock, receiving 80/4=20. Our debt is reduced to 15.

Stock goes up to 100 The new ϕ is $(20 - 0)/(120 - 80) = .5$. So we must buy 1/4 unit stock, spending 100/4=25. Our debt goes up to 40.

Stock goes down again to 80 We hold 40 dollars worth of stock and we owe 40. Our position is worth 40-40=0, replicating the option, which is worthless since it's out of the money. (Different viewpoint: the investment bank that sold the option can liquidate its position, selling the stock at market and using the proceeds to pay off the loan. This results in neither a loss nor a gain.)

Notice that the portfolio changes from one time to the next but the changes are *self-financing* – i.e. the total value of the portfolio before and after each trade are the same. (The investment bank neither receives or spends money except at the initial time, when it sells the option.)

Our example shows the importance of tracking ϕ, the number of units of stock to be held as you leave a given node. It characterizes the replicating portfolio (the "hedge"). Its value is known as the *Delta* of the claim. Thus:

$$\Delta_{\text{now}} = \text{our } \phi = \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}.$$

To understand the meaning Δ, observe that as you leave node j,

$$\text{value of claim at a node j} = \Delta_j s_j + b_j$$

by definition of the replicating portfolio (here b_j is the value of the bond holding at that moment). If the value of the stock changes by an amount ds while the bond holding stays fixed, then the value of the replicating portfolio changes by Δds. Thus Δ is a sort of derivative:

$$\Delta = \text{rate of change of replicating portfolio value, with respect to change of stock price.}$$

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Our valuation algorithm is easy to implement. But in the one-period setting we had more than an algorithm: we also had a *formula* for the value of the option, as the discounted expected value using a risk-neutral probability. A similar formula exists in the multiperiod setting. To see this, it is most convenient to work with a general binomial tree. Consider, for example, a tree with two time steps. The risk-neutral probabilities $q_j, 1-q_j$ are determined by the embedded one-period models. (Remember, the risk-neutral probabilities are characteristic of the market; they don't depend on the contingent claim under consideration.) In this case:

$$q_1 = \frac{e^{r\delta t}s_1 - s_2}{s_3 - s_2}, \quad q_2 = \frac{e^{r\delta t}s_2 - s_4}{s_5 - s_4}, \quad q_3 = \frac{e^{r\delta t}s_3 - s_6}{s_7 - s_6}.$$

As we work backward through the tree, we get a formula for the value of the contingent claim at each node, as a discounted weighted average of its values at maturity. In fact, writing $f(j)$ for the value of the contingent claim f at node j,

$$f(3) = e^{-r\delta t}[q_3 f(7) + (1 - q_3) f(6)]$$

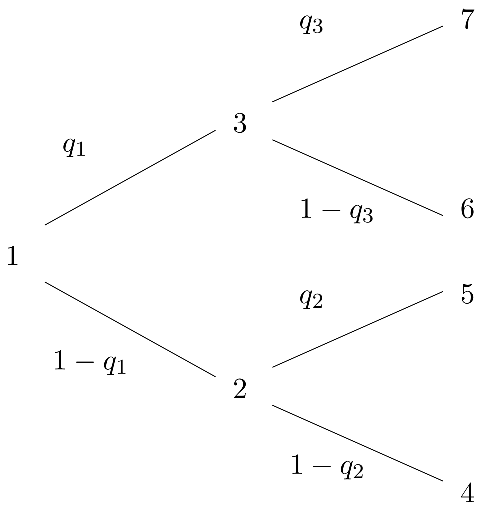

Figure 6: General binomial tree with two time steps.

and

$$f(2) = e^{-r\delta t}[q_2 f(5) + (1 - q_2) f(4)]$$

so

$$\begin{aligned} f(1) &= e^{-r\delta t}[q_1 f(3) + (1 - q_1) f(2)] \\ &= e^{-2r\delta t}[q_1 q_3 f(7) + q_1(1 - q_3) f(6) + (1 - q_1) q_2 f(5) + (1 - q_1)(1 - q_2) f(4)]. \end{aligned}$$

It should be clear now what happens, for a binomial tree with any number of time periods:

$$\text{initial value of the claim} = e^{-rN\delta t} \sum_{\text{final states}} [\text{probability of the associated path}] \times [\text{payoff of state}],$$

where the probability of any path is the product of the probabilities of the individual risk-neutral probabilities along it. (Thus: the different risk-neutral probabilities must be treated as if they described independent random variables.)

A similar rule applies to recombinant trees, since they are just special binomial trees in disguise. We must simply be careful to count the paths with proper multiplicities. For example, consider a two-period model with a recombinant tree and $s_{\text{up}} = u s_{\text{now}}$, $s_{\text{down}} = d s_{\text{now}}$. In this case the formula becomes

$$f(1) = e^{-2r\delta t}[(1 - q)^2 f(4) + 2q(1 - q) f(5) + q^2 f(6)]$$

with $q = (e^{r\delta t} - d)/(u - d)$, since there are two distinct paths leading to node 5.

The preceding calculation extends easily to recombinant trees with any number of time steps. The result is one of the most famous and important results of the theory: an explicit formula for the value of a European option. This is in a sense the binomial tree version of the Black-Scholes formula. (To really use it, of course, we'll need to know how to specify

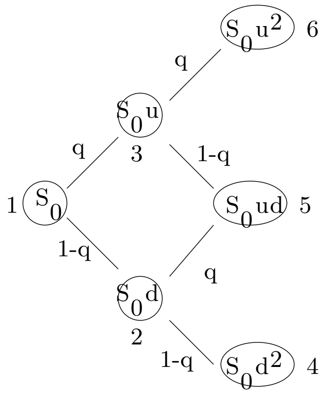

Figure 7: Price states in a multiplicative recombinant tree.

the parameters u and d; we'll come to that soon.) Consider an N-step recombinant stock price model with with $s_{\text{up}} = us_{\text{now}}$, $s_{\text{down}} = ds_{\text{now}}$, and s_0=initial spot price. Then the present value of an option with payoff $f(S_T)$ is

$$e^{-rN\delta t} \sum_{j=0}^{N} \left[\binom{N}{j} q^j (1-q)^{N-j} f(s_0 u^j d^{N-j}) \right].$$

with $q = (e^{r\delta t} - d)/(u-d)$. This holds because there are $\binom{N}{j}$ different ways of accumulating j ups and $N-j$ downs in N time-steps (just as there are $\binom{N}{j}$ different ways of getting heads exactly j times out of N coin flips.) Making this specific to European puts and calls: a call with strike price K has present value

$$e^{-rN\delta t} \sum_{j=0}^{N} \left[\binom{N}{j} q^j (1-q)^{N-j} (s_0 u^j d^{N-j} - K)_+ \right];$$

a put with strike price K has present value

$$e^{-rN\delta t} \sum_{j=0}^{N} \left[\binom{N}{j} q^j (1-q)^{N-j} (K - s_0 u^j d^{N-j})_+ \right].$$