Derivative Securities – Section 5
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The Black-Scholes formula and its applications. This Section deduces the Black-Scholes formula for a European call or put, as a consequence of risk-neutral valuation in the continuous time limit. Then we discuss the Delta, Gamma, Vega, and Rho of a portfolio, and their practical importance. Our treatment is closest to Jarrow and Turnbull, however Hull's treatment of this material is also excellent. We assume throughout that the underlying asset pays no dividend and has no carrying cost.

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The Black-Scholes formula for a European call or put. The upshot of Section 4 is this: the value at time t of a European option with payoff $f(s_T)$ is

$$V(f) = e^{-r(T-t)} E_{\mathrm{RN}}[f(s_T)].$$

Here $E_{\mathrm{RN}}[f(s_T)]$ is the expected value of the price at maturity with respect to a special probability distribution – the risk-neutral one. This distribution is determined by the property that

$$s_T = s_t \exp\left[(r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}Z\right]$$

where s_t is the spot price at time t and Z is Gaussian with mean 0 and variance 1. Equivalently: $\log[s_T/s_t]$ is Gaussian with mean $(r - \frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$.

This formula can be evaluated for any payoff f by numerical integration. But for special payoffs – including the put and the call – we can get explicit expressions in terms of the "cumulative distribution function"

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

($N(x)$ is the probability that a Gaussian random variable with mean 0 and variance 1 has value $\leq x$.) The explicit formulas have advantages over numerical integration: besides being easy to evaluate, they permit us to see quite directly how the value and the hedge portfolio depend on strike price, spot price, risk-free rate, and volatility.

It's sufficient, of course, to consider $t = 0$. Let

$$\begin{aligned} c[s_0, T; K] &= \text{value at time 0 of a European call with strike } K \\ &\quad\ \text{and maturity } T\text{, if the spot price is } s_0; \\ p[s_0, T; K] &= \text{value at time 0 of a European put with strike } K \\ &\quad\ \text{and maturity } T\text{, if the spot price is } s_0. \end{aligned}$$

The explicit formulas are:

$$\begin{aligned} c[s_0, T; K] &= s_0 N(d_1) - K e^{-rT} N(d_2) \\ p[s_0, T; K] &= K e^{-rT} N(-d_2) - s_0 N(-d_1) \end{aligned}$$

in which

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\log(s_0/K) + (r + \tfrac{1}{2}\sigma^2)T\right] \\ d_2 &= \frac{1}{\sigma\sqrt{T}}\left[\log(s_0/K) + (r - \tfrac{1}{2}\sigma^2)T\right] = d_1 - \sigma\sqrt{T}. \end{aligned}$$

To derive these formulas we use the following result. (The calculation at the end of Section 4 was a special case.)

Lemma: Suppose X is Gaussian with mean μ and variance σ^2. Then for any real numbers a and k,

$$E\left[e^{aX} \text{ restricted to } X \geq k\right] = e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)$$

with $d = (-k + \mu + a\sigma^2)/\sigma$.
Proof: The left hand side is defined by

$$E\left[e^{aX} \text{ restricted to } X \geq k\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_k^\infty e^{ax} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx.$$

Complete the square:

$$ax - \frac{(x-\mu)^2}{2\sigma^2} = a\mu + \tfrac{1}{2}a^2\sigma^2 - \frac{[x - (\mu + a\sigma^2)]^2}{2\sigma^2}.$$

Thus

$$E\left[e^{aX} \mid X \geq k\right] = e^{a\mu + \frac{1}{2}a^2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_k^\infty \exp\left[\frac{-[x - (\mu + a\sigma^2)]^2}{2\sigma^2}\right] dx.$$

If we set $u = [x - (\mu + a\sigma^2)]/\sigma$ and $\kappa = [k - (\mu + a\sigma^2)]/\sigma$ this becomes

$$\begin{aligned} e^{a\mu + \frac{1}{2}a^2\sigma^2} \cdot \frac{1}{\sqrt{2\pi}} \int_\kappa^\infty e^{-u^2/2}\, du &= e^{a\mu + \frac{1}{2}a^2\sigma^2}[1 - N(\kappa)] \\ &= e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d) \end{aligned}$$

where $d = -\kappa = (-k + \mu + a\sigma^2)/\sigma$.

We apply this to the European call. Our task is to evaluate

$$e^{-rT} \int_{-\infty}^\infty (s_0 e^x - K)_+ \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx.$$

The integrand is nonzero when $s_0 e^x > K$, i.e. when $x > \log(K/s_0)$. Applying the Lemma with $a = 1$ and $k = \log(K/s_0)$ we get

$$e^{-rT} \int_k^\infty s_0 e^x \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx = s_0 N(d_1);$$

applying the Lemma again with $a = 0$ we get

$$e^{-rT} \int_k^\infty K \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx = Ke^{-rT} N(d_2);$$

combining these results gives the formula for $c[s_0, T; K]$.

The formula for the value of a European put can be obtained similarly. Or – easier – we can derive it from the formula for a call, using put-call parity:

$$\begin{aligned} p[s_0, T; K] &= c[s_0, T; K] + Ke^{-rT} - s_0 \\ &= Ke^{-rT}[1 - N(d_2)] - s_0[1 - N(d_1)] \\ &= Ke^{-rT} N(-d_2) - s_0 N(-d_1). \end{aligned}$$

For options with maturity T and strike price K, the value at any time t is naturally $c[s_t, T - t; K]$ for a call, $p[s_t, T - t; K]$ for a put.

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Hedging. We know how to hedge in the discrete-time, multiperiod binomial tree setting: the payoff is replicated by a portfolio consisting of $\Delta = \Delta(0, s_0)$ units of stock and a (long or short) bond, chosen to have the same value as the derivative claim. At time δt the stock price changes to $s_{\delta t}$ and the value of the hedge portfolio changes by $\Delta(s_{\delta t} - s_0)$. The new value of the hedge portfolio is also the new value of the option, so

$$\Delta(0, s_0) = \frac{\text{change in value of option from time 0 to } \delta t}{\text{change in value of stock from time 0 to } \delta t}.$$

The replication strategy requires a self-financing trade at every time step, adjusting the amount of stock in the portfolio to match the new value of Δ.

In the real world prices are not confined to a binomial tree, and there are no well-defined time steps. We cannot trade continuously. So while we can pass to the continuous time limit for the value of the option, we must still trade at discrete times in our attempts to replicate it. Suppose, for simplicity, we trade at equally spaced times with interval δt. What to use for the initial hedge ratio Δ? Not being clairvoyant we don't know the value of the stock at time δt, so we can't use the formula given above. Instead we should use its continuous-time limit:

$$\Delta(0, s_0) = \frac{\partial(\text{value of option})}{\partial(\text{value of stock})}.$$

There's a subtle point here: if the stock price changes continuously in time, but we only rebalance at discretely chosen times $j\delta t$, then we cannot expect to replicate the option perfectly using self-financing trades. Put differently: if we maintain the principle that the value of the hedge portfolio is equal to that of the option at each time $j\delta t$, then our trades will no longer be self-financing. We will address this point soon, after developing the continuous-time Black-Scholes theory. We'll show then that (if transaction costs are ignored) the expected cost of replication tends to 0 as $\delta t \to 0$. (In practice transaction costs are *not* negligible; deciding when, really, to rebalance, taking into account transaction costs, is an important and interesting problem – but one beyond the scope of this course.)

For the European put and call we can easily get formulas for Δ by differentiating our expressions for c and p: at time T from maturity the hedge ratio should be

$$\Delta = \frac{\partial}{\partial s_0} c[s_0, T; K] = N(d_1)$$

for the call, and

$$\Delta = \frac{\partial}{\partial s_0} p[s_0, T; K] = -N(-d_1)$$

for the put. The "hard way" to see this is an application of chain rule: for example, in the case of the call,

$$\frac{\partial}{\partial s_0} c = N(d_1) + s_0 N'(d_1)\frac{\partial d_1}{\partial s} - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial s}.$$

But $d_2 = d_1 - \sigma\sqrt{T}$, so $\partial d_1/\partial s = \partial d_2/\partial s$; also $N'(x) = \frac{1}{\sqrt{2\pi}}\exp[-x^2/2]$. It follows with some calculation that

$$s_0 N'(d_1)\frac{\partial d_1}{\partial s} - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial s} = 0,$$

so finally $\partial c/\partial s_0 = N(d_1)$ as asserted. There is however an easier way: differentiate the original formula expressing the value as a discounted risk-neutral expectation. Passing the derivative under the integral, for a call with strike K:

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial s_0} e^{-rT} \int_{-\infty}^{\infty} (s_0 e^x - K)_+ \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx \\
&= e^{-rT} \int_{-\infty}^{\infty} \frac{\partial (s_0 e^x - K)_+}{\partial s_0} \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx \\
&= e^{-rT} \int_{\log(K/s_0)}^{\infty} e^x \frac{1}{\sigma\sqrt{2\pi T}} \exp\left[\frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx \\
&= N(d_1).
\end{aligned}$$

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Since we have explicit formulas for the value of a put or call, we can differentiate them to learn the dependence on the underlying parameters. Some of these derivatives have names:

Definition	**Call**	**Put**
Delta $= \frac{\partial}{\partial s_0}$	$N(d_1) > 0$	$-N(-d_1) < 0$
Gamma $= \frac{\partial^2}{\partial s_0^2}$	$\frac{1}{s_0 \sigma \sqrt{2\pi T}} \exp(-d_1^2/2) > 0$	$\frac{1}{s_0 \sigma \sqrt{2\pi T}} \exp(-d_1^2/2) > 0$
Theta $= \frac{\partial}{\partial t}$	$-\frac{s_0 \sigma}{2\sqrt{2\pi T}} \exp(-d_1^2/2) - rKe^{-rT} N(d_2) < 0$	$-\frac{s_0 \sigma}{2\sqrt{2\pi T}} \exp(-d_1^2/2) + rKe^{-rT} N(-d_2) \gtrless 0$
Vega $= \frac{\partial}{\partial \sigma}$	$\frac{s_0 \sqrt{T}}{\sqrt{2\pi}} \exp(-d_1^2/2) > 0$	$\frac{s_0 \sqrt{T}}{\sqrt{2\pi}} \exp(-d_1^2/2) > 0$
Rho $= \frac{\partial}{\partial r}$	$TKe^{-rT} N(d_2) > 0$	$-TKe^{-rT} N(-d_2) < 0.$

These formulas apply at time $t = 0$; the formulas applicable at any time t are similar, with T replaced by $T - t$. These are obviously useful for understanding how the value of the option changes with time, volatility, etc. But more: they are useful for designing improved hedges. For example, suppose a bank sells two types of options on the same underlying asset, with different strike prices and maturities. As usual the bank wants to limit its exposure to changes in the stock price; but suppose in addition it wants to limit its exposure to changes (or errors in specification of) volatility. Let $i = 1, 2$ refer to the two types of options, and let n_1, n_2 be the quantities held of each. (These are negative if the bank sold the options.) The bank naturally also invests in the underlying stock and in risk-free bonds; let n_s and n_b be the quantities held of each. Then the value of the bank's initial portfolio is

$$V_{\text{total}} = n_1 V_1 + n_2 V_2 + n_s s_0 + n_b.$$

We already know how the stock and bond holdings should be chosen if the bank plans to replicate (dynamically) the options: they should satisfy

$$V_{\text{total}} = 0$$

and

$$n_1 \Delta_1 + n_2 \Delta_2 + n_s = 0.$$

Notice that the latter relation says $\partial V_{\text{total}} / \partial s_0 = 0$: the value of the bank's holdings are insensitive (to first order) to changes in the stock price.

If we were dealing in just one option there would be no further freedom: we would have two homogeneous equations in three variables n_1, n_s, n_b, restricting their values to a line – so that n_1 determines n_s and n_b. That's the situation we're familiar with. But if we're dealing in two (independent) options then we have the freedom to impose one additional linear equation. For example we can ask that the portfolio be insensitive (to first order) to changes in σ by imposing the additional condition

$$n_1 \text{Vega}_1 + n_2 \text{Vega}_2 = 0.$$

Thus: by selling the two types of assets in the proper proportions the bank can reduce its exposure to change or misspecification of volatility.

If the bank sells three types of options then we have room for yet another condition – e.g. we could impose first-order insensitivity to changes in the risk-free rate r. And so on. It is not actually necessary that the bank use the underlying stock as one of its assets. Each option is *equivalent* to a portfolio consisting of stock and risk-free bond; so a portfolio consisting entirely of options and a bond position will function as a hedge portfolio so long as its total Δ is equal to 0.

Replication requires dynamic rebalancing. The bank must change its holdings at each time increment to set the new Δ to 0. In the familiar, one-option setting this was done by adjusting the stock and bond holdings, keeping the option holding fixed. In the present, two-option setting, maintaining the additional condition $\text{Vega}_{\text{total}} = 0$ will require the ratio between n_1 and n_2 to be dynamically updated as well, i.e. the bank will have to sell or buy additional options as time proceeds.