Derivative Securities – Section 7
Notes by Robert V. Kohn, Courant Institute of Mathematical Sciences.

Today's topic: some applications of the Black-Scholes PDE: (1) discrete-time hedging; (2) reduction to a constant-coefficient heat equation; and (3) pricing of barrier options. But first some notes:

Additions to the CIMS library reserve:

- I've asked that the book *Mathematics of Financial Derivatives* by S. Neftci (Academic Press, 1996) be added (this may take a few days). It gives a fuller introduction to stochastic differential equations, Ito's lemma, etc than any of the books presently on reserve.
- I've added to the Green Box a copy of the classic paper on binomial trees: *Option pricing: a simplified approach*, by J. Cox, S. Ross, and M. Rubinstein, J. Financial Econ. 7 (1979) 229-263. It's well worth reading. (My copy was taken from microfilm and it's highly imperfect – sorry!).
- I've added to the Green box copies of the papers *On the theory of perfect hedging*, by E. Omberg, Advances in Futures and Options Research 5 (1991) 1-29, and *Option pricing and replication with transaction costs*, by H.E. Leland, J. Finance 40 (1985) 1283-1301. Both address what happens to the Black-Scholes analysis when we remember that hedging must in practice be done at discrete times, not continuously in time. Omberg has a thoughtful discussion, capturing both mathematical and financial aspects, and pointing out many inaccuracies in the prior literature. Leland is more technical, and goes further, proposing a theory of option pricing in the presence of transaction costs.

Errors and typos in recent handouts:

- Problem 1a of HW4 asked you to show something that isn't true. Please interpret that problem as asking you to evaluate the hedge ratio (the "Delta") of a squared call, by differentiating under the integral then evaluating the resulting expression.
- The table in the Section 5 notes that gave formulas for "the Greeks" of a put and a call gave the wrong formula for Vega. The right formula is Vega $= \frac{s_0\sqrt{T}}{\sqrt{2\pi}} \exp(-d_1^2/2)$.

****

Discrete-time hedging. Suppose an investment bank sells an option and tries to replicate it dynamically, but the bank trades only at evenly spaced time intervals $j\delta t$. (Now δt is positive, not infinitesimal). The bank follows the standard trading strategy of rebalancing to hold $\phi = \partial V/\partial s$ units of stock each time it trades, where V is the value assigned by the Black-Scholes theory. As we shall see in a moment, this strategy is no longer self-financing – but it is *nearly so*, in a suitable stochastic sense, in the limit $\delta t \to 0$. My discussion follows the beginning of Leland's article; Omberg is also well worth reading.

People often ask, when examining the derivation of the Black-Scholes PDE by examination of the hedging strategy, "Why do we apply Ito's lemma to $V(s(t),t)$ but not to Δ, even though the choice of Δ also depends on $s(t)$?" The answer, of course, is that the hedge portfolio is held fixed from t to $t+\delta t$. The following discussion – in which δt is small but not infinitesimal – should help clarify this point.

OK, let's return to that investment bank. The question is: how much additional money will the bank have to spend over the life of the option as a result of its discrete-time (rather than continuous-time) hedging? We shall answer this by considering each discrete time interval, then adding up the results.

The bank holds a short position on the option and a long position in the replicating portfolio. The value of its position just after rebalancing at any time $t = j\delta t$ is (by hypothesis)

$$0 = -V(s(t),t) + \phi s(t) + [V(s(t),t) - \phi s(t)] = \text{short option} + \text{stock position} + \text{bond position}$$

with $\phi = \frac{\partial V}{\partial s}(s(t),t)$. The value of its position just before the next rebalancing is

$$-V(s(t+\delta t), t+\delta t) + \phi s(t+\delta t) + [V(s(t),t) - \phi s(t)]e^{r\delta t}.$$

The cost (or benefit) of rebalancing at time $t+\delta t$ is minus the value of the preceding expression. Put differently: it is the difference between the two preceding expressions. So it equals

$$\delta V - \phi\delta s - [V - \phi s](e^{r\delta t} - 1).$$

If we estimate δV by Taylor expansion keeping just the terms one normally keeps in Ito's lemma, we get (remembering that $\phi = \partial V/\partial s$)

$$\frac{\partial V}{\partial s}\delta s + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}(\delta s)^2 + \frac{\partial V}{\partial t}\delta t - \frac{\partial V}{\partial s}\delta s - rV\delta t + rs\frac{\partial V}{\partial s}\delta t.$$

Notice that the first and fourth terms cancel. Also notice that the substitution $(\delta s)^2 = \sigma^2 s^2 \delta t$ leads to an expression that vanishes, according to the Black-Scholes equation. Thus, the failure to be self-financing is attributable to two sources: (a) errors in the approximation $(\delta s)^2 \approx \sigma^2 s^2 \delta t$, and (b) higher order terms in the Taylor expansion. Our task is to estimate the associated costs.

Collecting the information obtained so far: if the investment bank re-establishes the "replicating portfolio" demanded by the Black-Scholes analysis at each multiple of δt then it incurs cost

$$\tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}(\delta s)^2 + \frac{\partial V}{\partial t}\delta t - rV\delta t + rs\frac{\partial V}{\partial s}\delta t$$

at each time step, plus an error of magnitude $|\delta t|^{3/2}$ due to higher order terms in the Taylor expansion. Using the Black-Scholes PDE, this cost has the alternative expression

$$\tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}[(\delta s)^2 - \sigma^2 s^2 \delta t] \quad \text{plus an error of order } |\delta t|^{3/2}.$$

It can be shown that since $ds = \sigma s dx + (\mu + \frac{1}{2}\sigma^2)s dt$,

$$\delta s = \sigma s u\sqrt{\delta t} + (\mu + \tfrac{1}{2}\sigma^2)s\delta t \quad \text{plus an error of order } |\delta t|^{3/2}$$

where u is Gaussian with mean 0 and variance 1 (c.f. our discussion of Ito's lemma). Therefore

$$(\delta s)^2 = \sigma^2 s^2 u^2 \delta t \quad \text{plus an error of order } |\delta t|^{3/2}.$$

Thus neglecting the error terms, the cost of refinancing at any given timestep is

$$\tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 (u^2 - 1)\delta t$$

where u is Gaussian with mean value 0 and variance 1. This expression is obviously random; its expected value is 0 and its standard deviation is of order δt. Moreover the contributions associated with different time intervals are independent. Notice that the distribution of refinancing costs is *not* Gaussian, since it is proportional to $u^2 - 1$ not u.

Pulling this together: since the expected value of $u^2 - 1$ is zero, the *expected cost* of refinancing at any given timestep is at most of order $|\delta t|^{3/2}$, due entirely to the "error terms." However the *actual cost* (or benefit) of refinancing is larger, a random variable of order δt. But the picture changes when we consider many time intervals. Over $n = T/\delta t$ intervals, the terms $\frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 (u^2 - 1)\delta t$ accumulate to a sum

$$\sum_{j=1}^{n} \tfrac{1}{2}\sigma^2 s^2(t_j)\frac{\partial^2 V}{\partial s^2}(s(t_j), t_j)(u_j^2 - 1)\delta t$$

with mean 0 and standard deviation of order $\sqrt{n\delta t^2} = \sqrt{T\delta t}$; the sum is still random, but it's small, statistically speaking, if δt is close to zero, by a sort of law-of-large-numbers. (Notice the resemblance of this argument to our explanation of Ito's lemma. That's no accident: we are in essence deriving Ito's formula all over again.) We've been ignoring the error terms – but they cause no trouble, because they too accumulate to a term of order $\sqrt{\delta t}$, because $n(\delta t)^{3/2} = T\sqrt{\delta t}$.

Final conclusion: the errors of refinancing tend to self-cancel, by a sort of law-of-large-numbers, since their mean value is 0. The net effect, when δt is small, is random but small — in the sense that its mean and standard deviation are of order $\sqrt{\delta t}$.

****

Using the Black-Scholes differential equation to derive explicit solution formulas. We now show how the Black-Scholes PDE can be reduced, by a convenient change of variables, to a constant-coefficient heat equation. Then we discuss the valuation of barrier options – whose valuation requires solving a boundary-value problem rather than the whole-space initial value problem. The book by Wilmott, Howison, Dewynne is a good source for this material. The newer book by Wilmott is good too.

Reduction to the heat equation by change of variables. The most basic parabolic PDE is the heat equation, $u_t = u_{xx}$. The Black-Scholes equation is really just this standard equation written in special variables. To see this, consider the preliminary change of variables $(s, t) \to (x, \tau)$ defined by

$$s = e^x, \quad \tau = \tfrac{1}{2}\sigma^2(T - t),$$

and let $v(x, \tau) = V(s, t)$. An elementary calculation shows that the Black-Scholes equation becomes

$$v_\tau - v_{xx} + (1-k)v_x + kv = 0$$

with $k = r/(\frac{1}{2}\sigma^2)$. We've done the main part of the job: reduction to a constant-coefficient equation. For the rest, consider $u(x, t)$ defined by

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

where α and β are constants. The equation for v becomes an equation for u, namely

$$(\beta u + u_\tau) - (\alpha^2 u + 2\alpha u_x + u_{xx}) + (1-k)(\alpha u + u_x) + ku = 0.$$

To get an equation without u or u_x we should set

$$\beta - \alpha^2 + (1-k)\alpha + k = 0, \quad -2\alpha + (1-k) = 0.$$

These equations are solved by

$$\alpha = \frac{1-k}{2}, \quad \beta = -\frac{(k+1)^2}{4}.$$

Thus,

$$u = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2\tau} v(x, \tau)$$

solves the linear heat equation $u_\tau = u_{xx}$. This can be used to give another proof of the integral formula for the value of an option (using the fundamental solution of the linear heat equation). It can also be used to understand the sense in which the value of an option at time $t < T$ is obtained by "smoothing" the payoff. (For this viewpoint to be useful you must know something about the behavior of the linear heat equation. Wilmott et al. have a good discussion of the basics. More can be found in any basic text on PDE's, for example F. John's book or the one by R. Guenther and J. Lee.)

Barrier options. A barrier option is like a European option *except* that it acquires or loses its value if the stock price goes above or below a specified barrier X:

An **up-and-in** option pays off only if the stock price crosses X from below prior to maturity.
A **down-and-in** option pays off only if the stock price crosses X from above prior to maturity.
An **up-and-out** option loses its value if the stock price crosses X from below prior to maturity.
A **down-and-out** option loses its value if the stock price crosses X from above prior to maturity.

These are our first examples of *path-dependent* options. They provide a nice example of the power of the Black-Scholes differential equation.

Let's concentrate on the case of a down-and-out call with strike price K, maturity T, and barrier X, with the further assumption that the barrier lies below both the spot price and the strike price. It pays $(s-K)_+$ if the stock price stays above X, but nothing if the stock price dips below X prior to maturity. We shall assume in the following that $X < K$.

Now the $V(s,t)$ solves the Black-Scholes equation in the restricted domain $s > X$, with boundary condition $V(X,t) = 0$, and final condition $V(s,T) = (s-K)_+$ at maturity (see the figure).

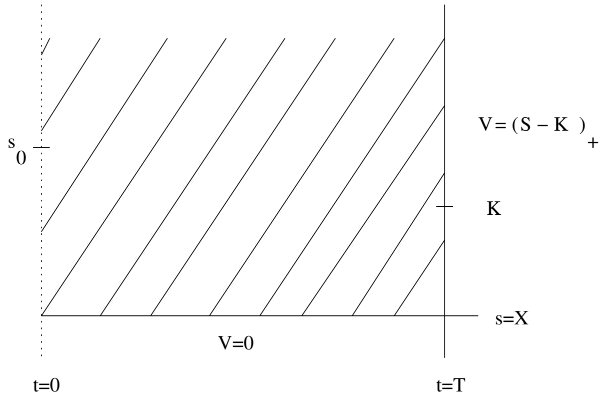

Figure 1: Boundary value problem for the down-and-out call.

One might expect we would have to leave it at that – a PDE in need of numerical solution. But actually there is an explicit formula:

$$V(s,t) = C(s,t) - \left(\frac{s}{X}\right)^{(1-k)} C(X^2/s, t)$$

where $k = r/(\frac{1}{2}\sigma^2)$ and $C(s,t)$ is the value of the ordinary European call with strike K and maturity T. It is a matter of arithmetic to check that this solves the PDE and the boundary conditions. But it's not so hard to understand where the formula comes from. Recall that under the change of variables

$$s = e^x, \quad \tau = \tfrac{1}{2}\sigma^2(T-t), \quad V(s,t) = e^{\alpha x + \beta\tau} u(x,\tau)$$

with $\alpha = (1-k)/2$, $\beta = -(k+1)^2/4$, the Black-Scholes PDE becomes the linear heat equation

$$u_\tau = u_{xx}.$$

Restricting $s > X$ is the same as restricting $x > \log X$, so the linear heat equation is to be solved for $x > \log X$, with $u = 0$ at $x = \log X$. Its initial value $u_0(x) = u(0,x)$, is obtained from the payoff of the call by change of variables: $u_0(x) = e^{-\alpha x}(e^x - K)_+$.

The trick for solving this linear heat conduction problem is standard: we extend the solution to $x < \log X$ by *odd reflection*, i.e. we look for a solution of $u_t = u_{xx}$ for all x, with the property that

$$u(x',t) = -u(x,t) \quad \text{when } x' \text{ is the reflection of } x \text{ about } \log X.$$

Such a solution must satisfy $u(\log X, t) = 0$, since the condition of odd symmetry gives $u(\log X, t) = -u(\log X, t)$.

Let's be more explicit about the condition of odd symmetry. Two points $x' < \log X < x$ are related by reflection about $\log X$ if $x - \log X = \log X - x'$, i.e. if $x' = 2\log X - x$. So a function $u(x,t)$ has odd symmetry about $\log X$ if it satisfies

$$u(2\log X - x, t) = -u(x,t) \quad \text{for all } x.$$

OK, the plan is clear: (a) Extend the initial data by odd symmetry about $x = \log X$; (b) solve the linear heat equation for $t > 0$ and all x; (c) observe that the resulting solution has odd symmetry for all $t > 0$, so its restriction to $x > \log X$ is the desired $u(x,t)$. Carrying out part (a): the desired initial data is $u_0(x) = e^{-\alpha x}(e^x - K)_+$ for $x > \log X$; moreover our assumption that $X < K$ assures that $u_0(\log X) = 0$. So the extended initial data is

$$f_0(x) = u_0(x) - u_0(2\log X - x)$$

with the convention that $u_0(x) = 0$ for $x < \log X$. Carrying out part (b): just solve the linear heat equation with initial data f_0; let's call the solution $f(x,t)$. Now part (c), the crucial assertion that $f(x,t)$ has odd symmetry about $x = \log X$. If not then $f(x,t)$ and $-f(2\log X - x, t)$ would be two *distinct* solutions of the heat equation with the *same* initial data. But the solution of the heat equation is unique – so $f(x,t) = -f(2\log X - x, t)$, as asserted. The restriction of f to $x > \log X$ is the solution u associated with the barrier option.

We can obtain an explicit formula for the value rather painlessly, by using the fact that our PDE's are linear. In fact: the value $V(s,t)$ of our barrier option is the difference of two terms. The first corresponds under the change of variables to

$$f_1(x,t) = \text{solution of the whole-space heat equation with initial data } u_0(x),$$

i.e. the first is the value $C(s,t)$ of the ordinary European call. (We note for later use the relation $f_1(x,\tau) = C(e^x, t)e^{-\alpha x - \beta\tau}$.) The second corresponds under the change of variables to

$$\begin{aligned} f_2(x,t) &= \text{solution of the whole-space heat equation with initial data } u_0(2\log X - x) \\ &= f_1(2\log X - x), \end{aligned}$$

so it is

$$\begin{aligned} e^{\alpha x + \beta\tau} f_2(x,\tau) &= e^{\alpha x + \beta\tau} f_1(2\log X - x, t) \\ &= e^{\alpha x + \beta\tau} e^{-(\alpha[2\log X - x] + \beta\tau)} C(e^{2\log X - x}, t) \\ &= X^{-2\alpha} s^{2\alpha} C(X^2/s, t) = (s/X)^{1-k} C(X^2/s, t). \end{aligned}$$

The solution formula asserted above is precisely the difference of these two terms.

This calculation is justified, of course, by a uniqueness theorem for the heat equation (or equivalently, a uniqueness theorem for the Black-Scholes equation). It says that there is exactly one solution $u(x, \tau)$ defined for $x \geq \log X$ and $t \geq 0$, with specified initial data $u(x, 0) = u_0(x)$ and specified boundary data $u(\log X, t) = 0$. Strictly speaking: we must also impose a growth condition at spatial infinity. It suffices to restrict attention to functions $u(x, \tau)$ which grow no faster than e^{cx} as $x \to \infty$, for some c. This condition is usually easy to justify for an option; for example, in the case of the down-and-out call we expect the value to be approximately that of a forward when $s \gg K$. So $V(s, t)$ grows linearly in s as $s \to \infty$, and $u(x, \tau)$ grows like $e^{(\alpha+1)x}$ as $x \to \infty$.

What about the down-and-in call? It has no value until the stock price crosses the barrier X from above; if that ever happens then it behaves like a standard call thereafter. No need for any hard work! It's obvious that

$$\text{down-and-out call} + \text{down-and-in call} = \text{standard call}$$

since the two portfolios have the same payoff. So the value of a down-and-in call is just the difference between the value of the standard call and the down-and-out call.