Derivative Securities – Section 8
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American options; continuous dividend yield. These notes show how the theory developed so far applies, with minor modifications, to (a) American options, and (b) options on stock indices or foreign currency. We continue to assume that the risk-free interest rate is constant – a reasonable approximation for options with relatively short maturities, on assets other than interest-based instruments.

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American options. (Chapter 7 of Jarrow-Turnbull provides a good introduction to this topic, at least in the setting of binomial trees.) Up to now we have discussed only European options, which (by definition) can only be exercised at a specified maturity date T. American options are different in that they permit early exercise: the holder of an American option can exercise it at any time up to the maturity T. Of the options actually traded in the market, the majority are American rather than European.

Clearly an American option is at least as valuable as the analogous European option, since the holder has the option to keep it to maturity.

Fact: An American call written on a stock that earns no dividend has the same value as a European call; early exercise is never optimal. To see why, suppose the strike price is K and consider the value of the American option "now," at some time $t < T$. Exercising the option now achieves a value at time t of $s_t - K$. Holding the option to maturity achieves a value at time t equal to that of a European call, $c[s_t, K, T-t]$. Without using the Black-Scholes formula (thus without assuming lognormal stock dynamics) we know the value of a European call is at least that of a forward with the same strike and maturity. Thus holding the option to maturity achieves a value at time t of at least $s_t - e^{-r(T-t)}K$. If $r > 0$ this is larger than $s_t - K$. So early exercise is suboptimal, as asserted.

The preceding is in some sense a fluke. When the underlying asset pays a dividend early exercise of a call can be optimal. But the simplest example where early exercise occurs is that of a put on a non-dividend-paying stock:

Fact: An American put written on a stock that earns no dividend can have a value greater than that of the associated European put; early exercise can be optimal. To see why, consider once again the value of the American option "now," at some time $t < T$. Exercising the option now achieves value $K - s_t$. Holding it to maturity achieves a value at time t equal to that of a European put, $p[s_t, K, T-t]$. Assuming lognormal stock price dynamics, p is given by the Black-Scholes formula, and its graph as a function of spot price s_t is shown in the figure.

The important point is that $p[s_t, K, T-t]$ is strictly less than $K - s_t$ when $s_t \ll K$. In such a situation we are better off to exercise at time t than to hold the option to maturity.

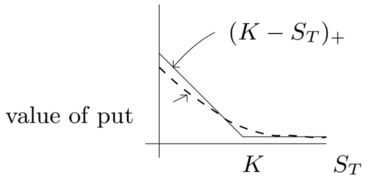

Figure 1: The value of a European put lies below the payoff when $s \ll K$.

(This does not show that exercise at time t is optimal, but it does show holding the option to maturity is suboptimal.)

For European options we have three different (but related) valuation techniques: (a) working backward through the binomial tree; (b) evaluating the discounted expected payoff (using the risk-neutral version of the price process); and (c) solving the Black-Scholes PDE. Each of the three viewpoints can be extended to American options. We assume for simplicity that the underlying asset pays no dividends.

Valuation using a binomial tree. This is perhaps the simplest approach, conceptually and numerically. We can use the same recombinant binomial tree as for a European option. (Remember: pricing is done using the risk-neutral process. If the underlying asset is lognormal with volatility σ then a convenient choice of the parameters defining the tree is $u = \exp[(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}]$, $d = \exp[(r - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}]$, where r is the risk-free rate). But since early exercise is permitted, we must ask at each node: is the option worth more "alive" or "dead"? If the option is worth more dead, then it should be exercised (by its holder) whenever the price arrives at that node. For example, consider the pricing of an American option with payoff $f(s)$ using a two-period recombinant tree:

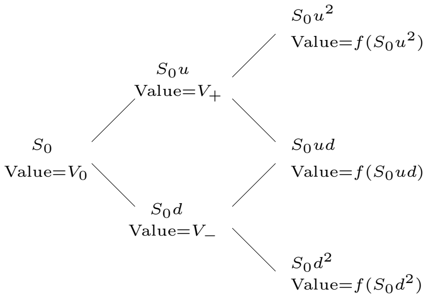

Figure 2: Valuation of an American option using a binomial tree.

When the stock price is s_0u the option is worth

$$f(s_0u) \text{ dead, and } e^{-r\delta t}[qf(s_0u^2) + (1-q)f(s_0ud)] \text{ alive.}$$

Allowing for both possibilities the value of the option at s_0u is

$$V_+ = \max\{f(s_0u), e^{-r\delta t}[qf(s_0u^2) + (1-q)f(s_0ud)]\}.$$

Similarly, when the stock price is $s_0 d$ the value is

$$V_- = \max\{f(s_0 d), e^{-r\delta t}[qf(s_0 ud) + (1-q)f(s_0 d^2)]\}.$$

The value at the initial time is obtained by repeating the process:

$$V_0 = \max\{f(s_0), e^{-r\delta t}[qV_+ + (1-q)V_-]\}.$$

Our example has only two time periods, but a binomial tree of any size is handled similarly.

Valuation using the discounted expected payoff. For a European option, we saw that the value assigned by the binomial tree was expressible in the form $e^{-rT} E_{\mathrm{RN}}[f(s(T))]$. A similar calculation applies to the American option – however $f(s(T))$ must be replaced by the value realized *at exercise*: the value of the option is $E_{\mathrm{RN}}\left[e^{-r\tau} f(s(\tau))\right]$ where τ is the exercise time. Once we've worked backward through the tree we know how to determine τ – for each realization of the risk-neutral process, it's the first time that realization reaches a node of the tree associated with early exercise (or T, if that realization does not reach an "early-exercise" node).

Actually, this viewpoint can also be used, at least conceptually, to *determine* the early-exercise criterion, without working backward through the tree. In fact,

$$\text{Value} = \max_{\text{exercise rules}} E_{\mathrm{RN}}\left[e^{-r\tau} f(s(\tau))\right].$$

In other words the exercise rule selected by backsolving the binomial tree is the one that maximizes the discounted expected payoff. An honest proof of this fact is not trivial – mainly because it requires formalization of what one means by an "exercise rule." But heuristically: any exercise rule determines a hedging strategy, i.e. a synthetic option that is available in the marketplace. So the max over exercise rules gives a lower bound for the value of the option. Our strategy of working backward through the tree gives an upper bound. The two bounds agree since the value obtained by working backward through the tree is associated with a special exercise rule.

Valuation using a PDE. (This material is not in Jarrow-Turnbull or Hull; you can find a brief summary in Wilmott or Wilmott-Howison-Dewynne.) For a European option the continuous-time limit of working backward through the tree amounts to solving the Black-Scholes PDE for $t < T$, with final data $f(s)$ at $t = T$. There is an analogous statement for an American option, however the PDE is replaced by a *free boundary problem*:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V}{\partial s} - rV \le 0,$$

$$V(s,t) \ge f(s),$$

and

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 + rs\frac{\partial V}{\partial s} - rV = 0 \quad \text{or} \quad V(s,t) = f(s).$$

The logic behind the first inequality is this: in our derivation of the Black-Scholes PDE, the crucial juncture was when we saw that the choice $\phi = \partial V/\partial s$ made $d(V - \phi s)$ deterministic:

$$d(V - \phi s) = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2} \frac{\partial^2 V}{\partial s^2} \sigma^2 s^2 \right) dt.$$

We concluded, by the principle of no arbitrage, that this must equal $r(V - \phi s)dt$. But that arbitrage argument assumed that you continued to hold the option. In the present context, where early exercise is permitted, the absence of arbitrage gives a weaker conclusion: the deterministic portfolio $(V - \phi s)$ can grow *no faster* than the risk-free rate. Thus

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2} \frac{\partial^2 V}{\partial s^2} \sigma^2 s^2 \right) \le r \left(V - \frac{\partial V}{\partial s} s \right);$$

this is our first inequality. The logic behind the second inequality is obvious: the value is no smaller than can be realized by immediate exercise. The third relation simply says that one of the first two relations always holds – because for any given (s, t) the optimal strategy involves either holding the option a little longer (in which case the Black-Scholes equation applies) or exercising it immediately.

We call this a free-boundary problem because the value is still governed by the Black-Scholes PDE in *some* region of the (s, t) plane – the region where immediate exercise isn't optimal – however this region isn't given as data but must be found as part of the problem. Schematically:

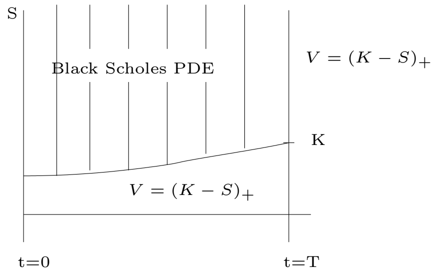

Figure 3: Schematic of the free boundary problem whose solution values an American put.

One can show that V and $\Delta = \partial V/\partial s$ are both continuous across the free boundary. Of course, on the "exercise" side of the boundary $V = f(s)$ and $\partial V/\partial s = f'(s)$ are known, giving two boundary conditions. If the domain of the PDE were known then just one boundary condition would be permitted; but the domain isn't known, and the extra boundary condition serves to fix the free boundary.

What if the underlying asset pays dividends at discrete times? The distribution of dividends (of predetermined sizes at predetermined times) is easily handled by minor modification of the techniques explained above. For an American call, exercise can only be optimal just before the distribution of a dividend. So we can value the option by working backward in time, using Black-Scholes or a (European) binomial tree to pass from one dividend date to the next, but taking the maximum of the value (a) if exercised, and (b) if not exercised, at each dividend date. For an American put we must still check for possible exercise at each time if using a tree – or we must still solve a free boundary problem between exercise dates if using the PDE. A point to watch out for: at the moment when a dividend is declared, the value of the underlying asset drops discontinuously by an amount equal to the dividend.

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Black-Scholes analysis with constant dividend yield. (This topic is treated in Jarrow-Turnbull chapters 11 and 12, and in Hull chapter 12.) We return to the case of a European option, and consider what happens when the underlying asset is a foreign currency or a stock index. These two cases are essentially identical (you learned this in Homework 1, problem 1): a foreign currency earns interest, while a stock index pays dividends (which we may choose to reinvest). Both cases are described, to a reasonable approximation, by supposing that the risky asset *pays dividends at a fixed rate D*. In other words, if you hold ϕ units of the risky asset now, and you do no trading, then you'll hold ϕe^{Dt} units after time t.

The bottom line is simple: options on such an asset can be priced using the Black-Scholes framework, however the risk-neutral process is different: it has drift $r-D-\frac{1}{2}\sigma^2$ rather than $r-\frac{1}{2}\sigma^2$. Less ambiguous, perhaps: the risk-neutral process solves the stochastic differential equation $ds = (r-D)sdt + \sigma sdw$ rather than $ds = rsdt + \sigma sdw$. We shall explain this assertion in two different ways: (a) using binomial trees, and (b) using the Black-Scholes PDE.

Using binomial trees. Suppose the subjective price process is lognormal, say $\log s(t) = \log s(0) + \mu t + \sigma w(t)$. (For foreign currency $s(t)$ is the exchange rate, in dollars per unit foreign currency; for a stock index $s(t)$ is the price *without* reinvestment of dividends). The value of the option at maturity T is a function of $s(T)$. Our strategy is the same as used earlier in the semester: choose a binomial tree that mimics the subjective price process; then find the appropriate risk-neutral probabilities; then value the option by working backward in the tree.

The constant dividend yield D changes only the middle part of this program – the formula for the risk-neutral probabilities. To see how, we need only consider a single time period.

Consider the portfolio consisting at time 0 of ϕ units of risky asset and ψ dollars worth of risk-free asset. Its initial value is $\phi s_0 + \psi$, and its value at the next time period is either $\phi s_0 u e^{D\delta t} + \psi e^{r\delta t}$ or $\phi s_0 d e^{D\delta t} + \psi e^{r\delta t}$, depending whether the risky asset went up or down. The situation is identical with that of a non-dividend-paying binomial market with $s_{\text{up}} = s_0 u e^{D\delta t}$ and $s_{\text{down}} = s_0 d e^{D\delta t}$. So by our previous analysis of (non-dividend-paying) binomial trees, the risk-free probability associated with the "up" state is

$$q = \frac{e^{r\delta t} - de^{D\delta t}}{ue^{D\delta t} - de^{D\delta t}} = \frac{e^{(r-D)\delta t} - d}{u-d},$$

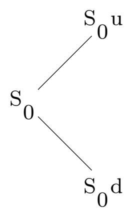

Figure 4: Standard one-period binomial tree.

and the value of the option at time 0 is

$$f_0 = e^{-r\delta t}[qf_{\text{up}} + (1-q)f_{\text{down}}]$$

where f_{up} and f_{down} are its values in the "up" and "down" states respectively. The extension to a multiperiod tree is obvious: we still work backward in the tree, and the value of the option is still the discounted expected payoff; the only difference is that we must use $q = (e^{(r-D)\delta t} - d)/(u-d)$ when we work backward in the tree. In the continuous-time limit: if the underlying asset has continuous dividend yield D then the value of a European option with payoff $f(s(T))$ is

$$e^{-rT}E[f(s_0e^X)] \quad \text{where } X \text{ is Gaussian with mean } (r - D - \tfrac{1}{2}\sigma^2)T \text{ and variance } \sigma^2 T.$$

The preceding argument shows, in essence, that the risk-neutral process solves the stochastic differential equation $ds = (r-D)sdt + \sigma sdw$. Here's an easy way to remember this. When there is no dividend, the risk-neutral process is $ds = rsdt + \sigma sdw$. It has has expected return r, in the sense that $(d/dt)E_{\text{RN}}[ds] = rs(t)$. When the asset pays continuous dividend yield D, the risk-neutral process has expected return $r - D$ *without counting the dividend yield*, so it has expected return r *if we include the dividend yield.*

An alternative analysis using the stochastic differential equation. We can obtain the same result by reconsidering the continuous-time hedging argument that led to the Black-Scholes PDE. Our lognormal hypothesis is equivalent to the stochastic differential equation

$$ds = \sigma sdx + (\mu + \tfrac{1}{2}\sigma^2)sdt, \quad s(0) = s_0.$$

Let $V(s(t), t)$ be the value of the option at stock price s and time t. Arguing as in Section 7, we consider a portfolio consisting of a short position in the option and a long position in the hedge portfolio (which consists of $\phi = (\partial V/\partial s)(s(t), t)$ units of stock, and $V - \phi s$ dollars risk-free). Its value at time t is

$$-V + \phi s + (V - \phi s) = 0$$

where $V = V(s(t), t)$ and $s = s(t)$; its value at time $t + \delta t$ is

$$-(V + \delta V) + \phi(s + \delta s + Ds\delta t) + (V - \phi s)(1 + r\delta t)$$

using the approximations $e^{D\delta t} \approx 1 + D\delta t$ and $e^{r\delta t} \approx 1 + r\delta t$. The only new term is the one associated with the dividends. To get to the PDE we must (a) use Ito's formula to estimate δV, then (b) set the value at time $t + \delta t$ to 0. Writing ds rather than δs, as we usually do for the Ito calculus, this gives

$$(V_t + V_s ds + \tfrac{1}{2}V_{ss}\sigma^2 s^2 dt) - (\phi ds + \phi D s dt) - (V - \phi s) r dt = 0$$

which becomes, after the subsitution $\phi = V_s$,

$$V_t + (r - D)sV_s + \tfrac{1}{2}\sigma^2 V_{ss} - rV = 0.$$

This is the Black-Scholes PDE for options on an asset with continuous dividend yield D. It is of course consistent with the one we obtained using binomial trees: the solution of this PDE (with $V(s,T) = f(s)$ at the final time T) satisfies

$$V(s(t), t) = e^{-r(T-t)} E_{\mathrm{RN}}[f(s(T)]$$

if we understand the risk-neutral process to be the one solving $ds = (r - D)sdt + \sigma s dw$. The proof is essentially the same as the argument we gave at the end of the Section 6 notes.

A shortcut to deriving explicit solutions. It is not necessary to derive new solution formulas. Instead we can use our existing solution formulas (derived for a non-dividend-paying asset) together with the following simple rule: *To value an option on an asset with continuous dividend yield D and maturity T, reduce the spot price by $e^{-D(T-t)}$ then apply the no-dividend-yield formula at this reduced spot price.* Thus, for example, the value at time 0 of a call with strike K is

$$s_0 e^{-DT} N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\log(s_0 e^{-DT}/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\log(s_0/K) + (r - D + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\log(s_0 e^{-DT}/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\log(s_0/K) + (r - D - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Similarly the value of a put is $Ke^{-rT}N(-d_2) - s_0 e^{-DT} N(-d_1)$.

The justification of this rule is easy. We may consider just $t = 0$. Starting from the valuation formula as the discounted risk-neutral expectation, we observe that if X Gaussian with mean $(r - D - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$ then

$$s_0 e^X = s_0 e^{(r - D - \frac{1}{2}\sigma^2)T + \sigma w(T)} = (s_0 e^{-DT}) e^Y$$

where Y has mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Thus the value of the option is

$$e^{-rT} E[f(s_0 e^X)] = e^{-rT} E[(f(s_0 e^{-DT} e^Y)]$$

and the right hand side is the "ordinary" (non-dividend-paying) Black-Scholes formula evaluated at the reduced spot price.

Of course we can also give a justification based on the Black-Scholes PDE. Let V solve the PDE derived above, and consider the change of variables

$$\bar{s} = se^{-D(T-t)}, \quad \bar{V}(\bar{s}, t) = V(s, t).$$

One verifies by an elementary calculation that $\bar{V}$ solves the no-dividend Black-Scholes equation. Moreover its final-time data is still the option payoff, since $\bar{s} = s$ at $t = T$. Since $\bar{s}(0) = e^{-DT}s(0)$, we conclude that

$$\text{Option value} = V(s(0), 0) = \bar{V}(e^{-DT}s(0), 0),$$

and the right hand side is again the "ordinary" (non-dividend-paying) Black-Scholes formula evaluated at the reduced spot price.

Black's formula. It's confusing to have so many different formulas. Fisher Black observed that the situation is simpler if we focus on the *forward price* rather than the spot price. Recall that if our underlying asset has spot price s_0 at time 0, then its *forward price* for delivery at time T is

$$F_0 = s_0 e^{(r-D)T}.$$

The value of an option with payoff $f(s(T)$ is evidently

$$e^{-rT}E[f(F_0 e^Z)] \quad \text{where } Z \text{ has mean } -\tfrac{1}{2}\sigma^2 T \text{ and variance } \sigma^2 T.$$

The advantage of this formula is that neither r nor D enters explicitly, except in the discount factor e^{-rT}. Instead, they are taken into account by use of the forward price F_0. Viewed this way, our formulas for the value of a call and a put become:

$$\text{call} = e^{-rT}[F_0 N(d_1) - KN(d_2)], \quad \text{put} = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

with

$$d_1 = \frac{\log(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(F_0/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

A final note. The financial intuition behind these results can be a bit confusing. It's natural to ask: why can't we just assume the dividends are reinvested, effectively changing the stochastic differential equation from $ds = (\mu + \frac{1}{2}\sigma^2)sdt + \sigma sdw$ to $ds = (\mu + \frac{1}{2}\sigma^2 + D)sdt + \sigma sdw$, then use the Black-Scholes framework (which is anyway insensitive to the drift)? The answer is this: we must be careful to recognize that the dividends are paid to those who hold the risky asset – but not to those who simply hold options on it.