Derivative Securities – Section 9
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Futures, and options on futures. Martingales and their role in option pricing. A brief introduction to stochastic interest rates. But first some comments about the final exam:

- The exam will be Tuesday December 19, in the normal class hour (7:10-9pm).
- I will distribute a short study guide (a list of topics to review) in the last lecture, Tuesday December 12. Part (possibly all) of the December 12 class will be devoted to answering questions and reviewing what we've covered.
- The exam will be closed-book, however you may bring two pages of notes (8.5×11, both sides, any font).
- The exam questions will test fundamental ideas and examples covered in the lectures and homeworks.

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Futures. We discussed *forward contracts* in detail in Section 1. A forward contract with delivery price K has payoff $s(T) - K$. We saw in Section 1 that value of this payoff is determined by arbitrage. No model of asset dynamics is needed, the market need not be complete, and no multiperiod trading is involved. In a constant interest rate environment the value of the forward at time 0 is $s_0 - Ke^{-rT}$. The hypothesis of constant interest rate is not really needed: the cash-and-carry argument involves no trading, so all that matters is the value at time 0 of a dollar received at time T. Denoting this by $B(0, T)$, the value of the forward contract is $s_0 - KB(0, T)$.

We also discussed the *forward price* in Section 1. It is the special delivery price for which the present value of the forward contract is 0. By elementary arithmetic, in a constant interest rate environment the forward price at time 0 is $F_0 = s_0e^{rT}$. Similarly the forward price at time t is $F_t = s_te^{r(T-t)}$. In a variable interest rate environment the forward price is $F_t = s_t/B(t, T)$.

A *futures contract* is similar to a forward – but not quite the same. Briefly, the investor who holds a long futures contract:

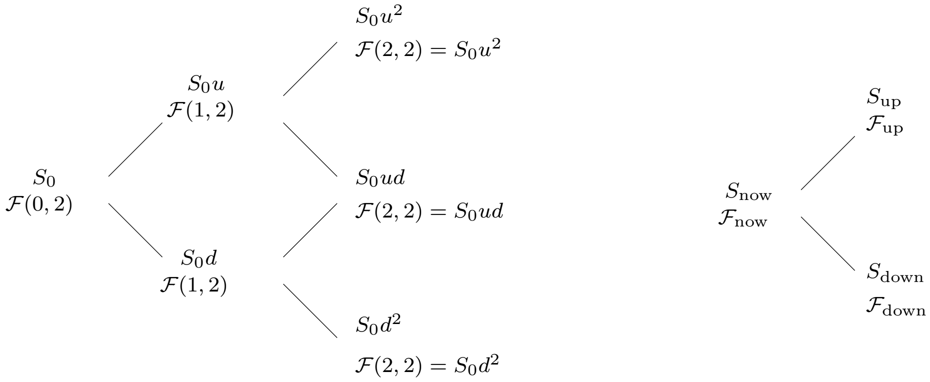
(a) pays nothing to acquire the contract – here it resembles a forward contract, with the forward price as delivery price;

(b) pays or receives funds daily as the value of the underlying asset varies – here it is quite different from a forward, where no cash flow occurs till maturity;

(c) buys the underlying asset at its market value s_T when the contract matures.

Thus the essential difference between a forward (with delivery price = forward price) and a future is that the settlement of the forward occurs entirely at maturity, while the settlement payoff of the future takes place daily. There are some other differences: futures are standardized, and they are bought and sold by financial institutions – thus they are liquid, traded instruments, which forwards are not. They are also often more liquid than the underlying asset itself. Thus while we often think of replicating an option by a (time-dependent) portfolio of the underlying asset and a cash account, in practice it is often better to use a (time-dependent) portfolio of futures on the underlying asset and a cash account.

Let us explain further how futures contracts work. Practical details aside (see Hull or Jarrow-Turnbull for those), this means explaining the *futures price* and the role it plays in *settlement*. We shall discuss this in the context of a binomial-tree market, following Section 5.7 of Jarrow-Turnbull. It's sufficient to consider the two-period tree in the figure: once we understand it, the multiperiod extension will be obvious. We write $\mathcal{F}(i, j)$ for the futures price at time i of a futures contract which matures at time j. Our goal is to understand a specific futures contract – maturing in this example at time 2 – so the maturity $j = 2$ will be fixed throughout our discussion; we shall determine its futures price $\mathcal{F}(i, 2)$ by working backward through the tree, starting at $i = 2$ and ending at $i = 0$.



As usual, the essential calculation involves a single-period binomial model branch. Suppose the stock price is s_{now} and it can go up to s_{up} or down to s_{down}. Suppose further that the futures price is already known to be $\mathcal{F}_{\text{up}}$ and $\mathcal{F}_{\text{down}}$ in the up and down states. To determine the futures price now, $\mathcal{F}_{\text{now}}$, let's look for a portfolio consisting of ϕ units of stock and ψ dollars risk-free that replicates the futures contract. The value of the futures contract now is 0, since one pays nothing to acquire a futures contract. Its value in the up state is $\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}$, and its value in the down state is $\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}$, since the settlement procedure involves a cash payment of $\mathcal{F}_{\text{new}} - \mathcal{F}_{\text{old}}$ at each time period. So the replicating portfolio must satisfy

$$\begin{aligned}
\phi s_{\text{now}} + \psi &= 0 \\
\phi s_{\text{up}} + \psi e^{r\delta t} &= \mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}} \\
\phi s_{\text{down}} + \psi e^{r\delta t} &= \mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}.
\end{aligned}$$

We know, from our treatment of binomial markets, that the last two equations alone give

$$\begin{aligned}\phi s_{\text{now}} + \psi &= e^{-r\delta t}[q(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}) + (1-q)(\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}})] \\ &= e^{-r\delta t}[q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}]\end{aligned}$$

where

$$q = \frac{e^{r\delta t}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$$

is the risk-neutral probability of the up state. Now the first equation (the condition that the futures contract have value 0 now) determines $\mathcal{F}_{\text{now}}$:

$$0 = \phi s_{\text{now}} + \psi = e^{-r\delta t}[q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}})]$$

whence

$$\mathcal{F}_{\text{now}} = q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}}.$$

This formula can of course be iterated over multiple time periods to give

$$\text{futures price at time } t = E_{\text{RN}}\left[\text{futures price at time } t'\right]$$

for any $t' > t$. At the time when the futures contract matures, its price is (by definition) the spot price of the underlying asset.

For a standard muliplicative binomial tree in a constant interest rate environment the risk-neutral probability is $q = (e^{r\delta t} - d)/(u-d)$, the same at every branch. Thus for the two-period tree shown in the preceding figure

$$\mathcal{F}(1,2) = \begin{cases} qs_0u^2 + (1-q)s_0ud & \text{if the price is } s_0u \\ qs_0ud + (1-q)s_0d^2 & \text{if the price is } s_0d \end{cases}$$

and

$$\mathcal{F}(0,2) = q^2s_0u^2 + 2q(1-q)s_0ud + (1-q)^2s_0d^2.$$

If the risk-free rate is constant then the futures price is equal to the forward price. This is true for any market (see Appendix 3B of Hull for a proof). In the special case of a binomial market it follows easily from the results just derived, together with the crucial feature of the risk-neutral probabilities that

$$s_0 = e^{-rT}E_{\text{RN}}\left[s(T)\right]$$

(this was clear from Section 3, and explicit at the beginning of Section 4). Using these facts: the futures price at time 0 for contracts maturing at time T is

$$\mathcal{F}_0 = E_{\text{RN}}\left[s(T)\right] = e^{rT}s_0$$

which is precisely the forward price.

Why have we fussed so much over futures prices, if in the end they are simply the same as forward prices? The answer is two-fold: (a) to clarify the essential nature of a futures contract (in particular its periodic settlement); (b) to prepare for modeling of interest-based instruments, and non-constant *stochastic* risk-free rates.

Notice the similarity – and the difference – between the results

$$\text{futures price at time } 0 = E_{\text{RN}}\,[\text{futures price at } t]$$

for a futures contract maturing at time $T > t$, and

$$\text{value at time } 0 = e^{-rt} E_{\text{RN}}\,[\text{value at time } t]$$

for a tradeable asset such as the underlying stock or any option. People sometimes get confused, and ask why there is no factor of e^{-rt} in the formula for the futures price. The answer is that these two formulas are intrinsically different. The futures price is not the value of a tradeable asset.

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Options on futures. Now consider a European option with maturity T, whose payoff is a function of a futures price $\mathcal{F}(T, T')$ for some $T' > T$. A typical example – a call with strike K – would give its holder the right to receive, at time T, one futures contract with maturity T', plus $\mathcal{F}(T, T') - K$ in cash. This instrument has payoff $(\mathcal{F}(T, T') - K)_+$, since the futures contract itself has value 0.

Options on futures are attractive because they involve only delivery of futures contracts and cash upon exercise – nobody has to purchase or sell the underlying asset. This is convenient, since the futures contracts are usually more liquid than the underlying asset. Options are traded on many kinds of futures, but options on interest rate futures have particularly active markets. We'll nevertheless continue to assume that the risk-free rate is constant for the moment, returning to the case of options on interest rate futures in a future lecture.

If the payoff of an option is determined by a futures price, then it's natural to value the option using the tree of futures prices rather than the tree of stock prices. As usual the value of the option is determined by working backward in the tree, so the heart of the matter is the handling of a single-period binomial market. Suppose the futures price now is $\mathcal{F}_{\text{now}}$ and at the next period the futures prices are $\mathcal{F}_{\text{up}}$ and $\mathcal{F}_{\text{down}}$. Assume further that the option's value at the next period is already known to be f_{up} in the up state, and f_{down} in the down state.

contract value $= \mathcal{F}_{\text{up}} - \mathcal{F}_0$
option value $= f_{\text{up}}$

contract value $= 0$
option value $= f_{\text{now}}$

contract value $= \mathcal{F}_{\text{down}} - \mathcal{F}_0$
option value $= f_{\text{down}}$

Rather than replicate the option using the underlying asset, it's convenient to replicate it using futures contracts. The value of the futures contract upon entry is 0, and its value at the next time period is $\mathcal{F}_{\rm up} - \mathcal{F}_{\rm now}$ or $\mathcal{F}_{\rm down} - \mathcal{F}_{\rm now}$ due the settlement procedure, so the relevant price tree is as shown in the figure.

Our results on binomial trees determine the option price as

$$f_{\rm now} = e^{-r\delta t}\left[pf_{\rm up} + (1-p)f_{\rm down}\right]$$

where p is the relevant risk-neutral probability, determined by

$$0 = e^{-r\delta t}\left[p(\mathcal{F}_{\rm up} - \mathcal{F}_{\rm now}) + (1-p)(\mathcal{F}_{\rm down} - \mathcal{F}_{\rm now})\right].$$

The last relation amounts to $\mathcal{F}_{\rm now} = p\mathcal{F}_{\rm up} + (1-p)\mathcal{F}_{\rm down}$, so we recogize that $p = q$ is the *same* risk-neutral probability we used to determine the futures prices. However let us forget this fact for a moment, and consider pricing the option using *only* the tree of forward prices. Then the convenient definition of p is

$$p = \frac{\mathcal{F}_{\rm now} - \mathcal{F}_{\rm down}}{\mathcal{F}_{\rm up} - \mathcal{F}_{\rm down}},$$

and the option price is determined by

$$f_{\rm now} = e^{-r\delta t}\left[pf_{\rm up} + (1-p)f_{\rm down}\right].$$

Working backward in the tree, we obtain (if the interest rate is constant) that the option value is its discounted expected payoff:

$$\text{option value at time } 0 = e^{-rT}E_{\rm RN}\left[\text{payoff at time } T\right].$$

Let's compare this result to the one obtained long ago for pricing ordinary options on lognormal assets. There the option price was the discounted risk-neutral expected value, using risk-neutral measure $\pi = \frac{e^{r\delta t}s_{\rm now} - s_{\rm down}}{s_{\rm up} - s_{\rm down}}$. Here the option price is the discounted risk-neutral value, using risk-neutral measure $p = \frac{\mathcal{F}_{\rm now} - \mathcal{F}_{\rm down}}{\mathcal{F}_{\rm up} - \mathcal{F}_{\rm down}}$. We see that the two situations are parallel, except that we must set $r = 0$ in the definition of the risk-neutral probability. Passing as usual to the continuous-time limit, we conclude that *an option on a futures contract can be valued using Black's formula*: if the futures price $\mathcal{F}_t$ is lognormal with volatility σ then an option with maturity T and payoff $f(\mathcal{F}_T)$ has value

$$e^{-rT}E[f(\mathcal{F}_0 e^Z)] \quad \text{where } Z \text{ has mean } -\tfrac{1}{2}\sigma^2 T \text{ and variance } \sigma^2 T.$$

In particular, for a call (payoff $(\mathcal{F}_T - K)_+$) or a put (payoff $(K - \mathcal{F}_T)_+$) we get the value

$$\text{call} = e^{-rT}[F_0 N(d_1) - KN(d_2)], \quad \text{put} = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

with

$$d_1 = \frac{\log(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(F_0/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Comparing these results with those of Section 8, we see that pricing an option on a futures price is analogous to pricing an option on an asset with continuous dividend yield equal to the risk-free rate.

We avoided making reference to the original stock price tree in the preceding argument. In particular we avoided using the fact that when interest rates are constant, forward and future prices are the same: $\mathcal{F}_t = F_t = s_t e^{r(T-t)}$. But we could alternatively have based our analysis on this fact. Indeed, it easily implies the crucial relation

$$\frac{\mathcal{F}_{\text{now}} - \mathcal{F}_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}} = \frac{e^{r\delta t} s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$$

from which all else follows. (In our prior notation this relation says that $p = q$.)

There is of course an alternative route to Black's formula using stochastic PDE's. Let us stop distinguishing between the forward price F and the futures price $\mathcal{F}$, since this discussion is restricted to the constant-interest-rate environment where they are the same. The critical assertion is that if $V(F(t), t)$ is the value of the option as a function of futures price, then V solves $V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} - rV = 0$ with final value $f(F)$, where f is the payoff. (This is equivalent to the assertion that $V(F_0, 0) = e^{-rT} E\left[f(F(T))\right]$ where $dF = \sigma F dw$, and its solution is given by Black's formula.) Arguing as in the last section, consider a portfolio consisting of a short position in the option and a long position in the hedge portfolio (which consists of $\phi = (\partial V/\partial F)(F(t), t)$ futures and $V(F(t), t)$ dollars risk-free). Its value at time t is

$$-V + \phi \cdot 0 + V = 0$$

since a futures contract costs nothing at the time of acquisition. Its value at time $t + \delta t$ is

$$-(V + \delta V) + \phi \delta F + V(1 + r\delta t);$$

the middle term $\phi \delta F$ represents the cost of settlement at time $t + \delta t$. To get the PDE we must (a) use Ito's formula to evaluate δV, then (b) set the value at time $t + \delta t$ to 0. We suppose the futures price solves $dF = \mu F dt + \sigma F dw$ for some μ. Then steps (a) and (b) lead to

$$(V_t dt + V_F dF + \tfrac{1}{2} V_{FF} \sigma^2 F^2 dt) - V_F dF - rV dt.$$

The dF terms cancel, and what remains is the desired relation

$$V_t + \tfrac{1}{2}\sigma^2 F^2 V_{FF} - rV = 0.$$

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Martingales. The basic prescription for working backward in a binomial tree is this: if f is the value of a tradeable security (such as an option) then

$$f_{\text{now}} = e^{-r\delta t}[q f_{\text{up}} + (1-q) f_{\text{down}}] = e^{-r\delta t} E_{\text{RN}}[f_{\text{next}}]$$

and if $\mathcal{F}$ is the futures price of a tradeable security then

$$\mathcal{F}_{\text{now}} = [q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}}] = E_{\text{RN}}[\mathcal{F}_{\text{next}}],$$

where q is the risk-neutral probability, defined by

$$s_{\text{now}} = e^{-r\delta t}[qs_{\text{up}} + (1-q)s_{\text{down}}] = e^{-r\delta t}E_{\text{RN}}[s_{\text{next}}].$$

When the risk-free rate is constant the factors of $e^{-r\delta t}$ don't bother us – we just bring them out front. When the risk-free rate is stochastic, however, we must handle them differently. To this end it is convenient to introduce a *money market account* which earns interest at the risk-free rate. Let $A(t)$ be its balance, with $A(0) = 1$. In the constant interest rate setting obviously $A(t) = e^{rt}$; in the variable interest rate setting we still have $A(t + \delta t) = e^{r\delta t}A(t)$, however r might vary from time to time, and even (if interest rates are stochastic) from one binomial subtree to another. With this this convention, the prescription for determining the price of a tradeable security becomes

$$f_{\text{now}}/A_{\text{now}} = E_{\text{RN}}[f_{\text{next}}/A_{\text{next}}]$$

since $A_{\text{now}}/A_{\text{next}} = e^{-r\delta t}$ where r is the risk-free rate. (This relation is valid even if the risk-free rate varies from one subtree to the next). Working backward in the tree, this relation generalizes to one relating the option value at any pair of times $0 \le t < t' \le T$:

$$f(t)/A(t) = E_{\text{RN}}[f(t')/A(t')].$$

Here, as usual, the risk-neutral expectation weights each state at time t' by the probability of reaching it via a coin-flipping process starting from time t — with independent, biased coins at each node of the tree, corresponding to the risk-neutral probabilities of the associated subtrees.

The preceding results say, in essence, that certain processes are *martingales*. Concentrating on binomial trees, a "process" is just a function g whose values are defined at every node. A process is said to be a *martingale* relative to the risk-neutral probabilities if it satisfies

$$g(t) = E_{\text{RN}}[g(t')]$$

for all $t < t'$. The risk-neutral probabilities are determined by the fact that

- $s(t)/A(t)$ is a martingale relative to the risk-neutral probabilities

where $s(t)$ is the stock price process. Option prices are determined by the fact that

- $f(t)/A(t)$ is a martingale relative to the risk-neutral probabilities

if f is the value of a tradeable asset. Futures prices are determined by the fact that

- $\mathcal{F}(t)$ is a martingale relative to the risk-neutral probabilities.

One advantage of this framework is that it makes easy contact with the continuous-time theory. The central connection is this: in continuous time, the solution of a stochastic differential equation $dy = f dt + g dw$ is a martingale if $f = 0$. Indeed, the expected value of a dw-stochastic integral is 0, so for any $t < t'$ we have $E[y(t)] = E\left[\int_t^{t'} f(\tau)\,d\tau\right] = \int_t^{t'} E[f(\tau)]\,d\tau$; for the right hand side to vanish (for all $t < t'$) we must have $E[f] = 0$. If f is deterministic then this condition says simply that $f = 0$.

We can use this insight to explain and/or confirm some results previously obtained by other means. We return here to the constant-interest-rate environment, so $A(t) = e^{rt}$.

Question: why does the risk-neutral stock price process satisfy $ds = rsdt + \sigma sdw$? Answer: because the risk-neutral stock price has the property that $s(t)/A(t) = s(t)e^{-rt}$ is a martingale. Explanation: if we assume that the risk-neutral price process has the form $ds = fdt + gdw$ for some f, we easily find that

$$d(se^{-rt}) = e^{-rt}ds - re^{-rt}sdt = (f - rs)dt + e^{-rt}gdw.$$

So se^{-rt} is a martingale exactly if $f = rs$. (You may wonder why the risk-neutral stock price process has the same *volatility* as the subjective stock price process. This is because changing the drift has the effect of re-weighting the probabilites of paths, without actually changing the set of "possible" paths; changing the volatility on the other hand has the effect of considering an entirely different set of "possible paths." This is the essential content of Girsanov's theorem, which is discussed and applied in the course Continuous-Time Finance.)

Question: why does the option price satisfy the Black-Scholes PDE? Answer: because the option price normalized by $A(t)$ must be a martingale. Explanation: suppose the option price has the form $V(s(t), t)$ for some function $V(s, t)$. Then

$$\begin{aligned} d\left(V(s(t), t)e^{-rt}\right) &= e^{-rt}dV - re^{-rt}Vdt \\ &= e^{-rt}(V_t dt + V_s ds + \tfrac{1}{2}V_{ss}\sigma^2 s^2 dt) - re^{-rt}Vdt \\ &= e^{-rt}(V_t + rsV_s + \tfrac{1}{2}\sigma^2 s^2 V_{ss} - rV)dt + e^{-rt}\sigma sV_s dw. \end{aligned}$$

For this to be a martingale the coefficient of dt must vanish. That is exactly the Black-Scholes PDE.

Question: why does the solution of the Black-Scholes PDE give the discounted expected payoff of the option? Answer: because the option price normalized by $A(t)$ is a martingale. Explanation: suppose V solves the Black-Scholes PDE, with final value $V(s, T) = f(s)$. We have shown that $e^{-rt}V(s(t), t)$ is a martingale. Therefore

$$V(s(0), 0) = E_{\mathrm{RN}}\left[e^{-rt}V(s(t), t)\right]$$

for any $t > 0$. Bringing e^{-rt} out of the expectation and setting $t = T$ gives

$$V(s(0), 0) = e^{-rT}E_{\mathrm{RN}}\left[V(s(T), T)\right] = e^{-rT}E_{\mathrm{RN}}\left[f(s(T))\right]$$

as asserted.

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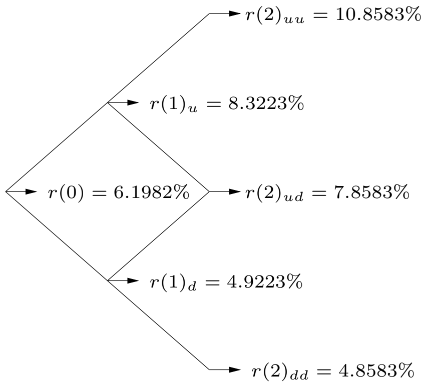
A brief introduction to stochastic interest rates. When considering interest-based instruments, the essential source of randomness is the interest rate itself. Let us briefly explain how the binomial-tree framework can be used for modeling stochastic interest rates,



following Jarrow-Turnbull Section 15.2. The basic idea is shown in the figure: each node of the tree is assigned a risk-free rate, different from node to node; it is the one-period risk-free rate for the binomial subtree just to the right of that node.

What probabilities should we assign to the branches? It might seem natural to start by figuring out what the subjective probabilities are. But why bother? All we really need for option pricing are the risk-neutral probabilities. Moreover we know (from Homework 3) that there is some freedom in the choice of the risk-neutral probability q, and that for lognormal dynamics it is always possible to set $q = 1/2$. So the usual practice is to

- restrict attention to the risk-neutral interest rate process.
- assume the risk-neutral probability is $q = 1/2$ at each branch, and
- choose the interest rates at the various nodes so that the long-term interest rates associated with the tree match those observed in the marketplace.

The last bullet – the *calibration* of the tree to market information – is of course crucial. We'll return to it in a few weeks. For now let's just be sure we understand what it means. In other words let's be sure we understand how such a tree determines long-term interest rates. As an example let's determine $B(0,3)$, the value at time 0 of a dollar received at time 3, for the tree shown in the figure. (Put differently: $B(0,3)$ is the price at time 0 of a zero-coupon bond which matures at time 3.) We take the convention that $\delta t = 1$ for simplicity.

Consider first time period 2. The value at time 2 of a dollar received at time 3 is $B(2,3)$; it has a different value at each time-2 node. These values are computed from the fact that

$$B(2,3) = e^{-r\delta t}[\tfrac{1}{2}B(3,3)_{\text{up}} + \tfrac{1}{2}B(3,3)_{\text{down}}] = e^{-r\delta t}$$

since $B(3,3) = 1$ in every state, by definition. Thus

$$B(2,3) = \begin{cases} e^{-r(2)_{uu}} = .897104 & \text{at node } uu \\ e^{-r(2)_{ud}} = .924425 & \text{at node } ud \\ e^{-r(2)_{dd}} = .952578 & \text{at node } dd. \end{cases}$$

Now we have the information needed to compute $B(1,3)$, the value at time 1 of a dollar received at time 3. Applying the rule

$$B(1,3) = e^{-r\delta t}[\tfrac{1}{2}B(2,3)_{\text{up}} + \tfrac{1}{2}B(2,3)_{\text{down}}]$$

at each node gives

$$B(1,3) = \begin{cases} e^{-r(1)_u}(\frac{1}{2}\cdot .897104 + \frac{1}{2}\cdot .924425) = .838036 & \text{at node } u \\ e^{-r(1)_d}(\frac{1}{2}\cdot .924425 + \frac{1}{2}\cdot .952578) = .893424 & \text{at node } d. \end{cases}$$

Finally we compute $B(0,3)$ by applying the same rule:

$$\begin{aligned} B(0,3) &= e^{-r\delta t}[\tfrac{1}{2}B(1,3)_{\text{up}} + \tfrac{1}{2}B(1,3)_{\text{down}}] \\ &= e^{-r(0)}[\tfrac{1}{2}\cdot .838036 + \tfrac{1}{2}\cdot .893424] = .8137. \end{aligned}$$