Derivative Securities – Course review and exam study guide
Distributed 12/12/00

1. One-period market models

   - Arbitrage-based pricing: value of a forward contract on a non-dividend-paying stock; value of a forward contract for foreign currency; put-call parity; etc.
   - Binomial market: value any contingent claim by finding a hedge portfolio. Expression for the value as the discounted risk-neutral expectation. The risk-neutral probability is uniquely determined by the condition that it give the right value for a forward.
   - Trinomial and more general markets: we can give upper and lower bounds for the value of a contingent claim by solving a pair of linear programming problems. The dual problem is an optimization over risk-neutral probabilities. A contingent claim is replicatable exactly if the upper and lower bounds coincide.

   Sample exam questions: value a forward contract by identifying a portfolio of known value with the same payoff at time T. (Like HW1, problems 1 and 3). Or take advantage of a mispriced forward – how much can you gain through this arbitrage? (Like HW1, problem 2). Consider a certain market with two assets and three final states (more details would be specified); which contingent claims are replicatable?

2. Multiperiod binomial trees

   - Valuing contingent claims: working backward in the tree.
   - Hedging: dynamic replication of a contingent claim.
   - The valuation formula: for a European option, value = discounted risk-neutral expectation of payoff at maturity.
   - American options: check at each node for the possibility of early exercise.
   - Continuous dividend yield, foreign currency, options on futures: similar framework, but the formula for the risk-neutral probability is different
   - Passage to the continuous-time limit: application of the central limit theorem. (Don't confuse the subjective and risk-neutral processes.)

   Sample exam questions: Value a contingent claim by working backward in a tree. Specify the hedging strategy, including rebalancing (like the example in Section 3, or HW3 problem 1). What if the option is American, i.e. permits early exercise? (Discussed in Section 8 and HW5, problems 4, 5). When considering an option on foreign currency, what is the proper choice of the risk-neutral probability and why? (Section 8 material again.) Consider an N-step multiplicative tree with $u = \exp(\mu\delta t + \sigma\sqrt{\delta t})$ and $d = \exp(\mu\delta t - \sigma\sqrt{\delta t})$; if the probability of going up is p and the probability of going down is $1 - p$, find the mean and variance of $\log s(N\delta t)$.

3. Derivation and use of the Black-Scholes formulas

- I'll give you the formula for $E[e^{aX}$ restricted to $X \geq k]$ if you need it. But you'll need to know how to choose the mean and variance of X, what to use for a, etc. for a specific application.
- Option values: deriving the BS formula for valuing a put or a call; interpretations of the terms; analogous formulas for other options such as a powered call.
- Hedging: deriving the formulas for Delta, Vega, etc.
- Qualitative properties, for puts and calls. Early exercise can be optimal for an American put.
- Continuous dividend yield, options on futures: Black's formula

Sample exam questions: for the risk-neutral price process, find the probability that its value at time T is greater than K. Derive a formula for the value or the Delta of a particular option (like HW3 problem 6, valuing an option with payoff s_T^n; or HW4 problem 1, concerning an option with payoff $(s_T - K)_+^2$).

4. The basic continuous-time theory

- Stochastic differential equations. The lognormal stock process as a special case.
- Applications of Ito's formula. Consequences of the fact that dw integrals have mean value 0 (are martingales).
- Derivation of the Black-Scholes PDE based on hedging and rebalancing.
- Interpretation of the Black-Scholes PDE: $e^{-rt}V(s(t), t)$ is a martingale for the risk-neutral price process, which solves $ds = rsdt + \sigma sdw$.
- Equivalence of pricing based on the solution formula (value = discounted expected payoff, using the risk-neutral probabilities) and based on solving the Black-Scholes PDE (value = $V(s_0, 0)$ where V solves the PDE with the option payoff as final-time at $t = T$.)

Sample exam questions: show that if $ds = rsdt + \sigma sdw$ then $\log s(t)$ is Gaussian with mean $r - (1/2)\sigma^2$ and variance $\sigma^2 t$. Show that the solution of $ds = rsdt + \sigma sdw$ has the property that $s(t)e^{-rt}$ is a martingale, i.e. $E[s(t)e^{-rt}]$ is independent of time; show further that if $V(s, t)$ solves the Black-Scholes differential equation then $V(s(t), t)e^{-rt}$ is a martingale, i.e. $E[V(s(t), t)e^{-rt}]$ is independent of time; use this to connect our two methods for finding the value of an option (by solving the Black-Scholes PDE, and by evaluating the discounted expected payoff using the risk-neutral process.) Derive the Black-Scholes PDE by considering a suitable hedging strategy.

5. Further continuous-time theory

- Equivalence of the Black-Scholes PDE and the linear heat equation
- Valuation formulas for Barrier options
- American options

Sample exam questions: show that early exercise is never optimal, for an American call on a non-dividend-paying stock. Show that it can be optimal for an American put. Show that it can be optimal on an American call, if the underlying has continuous dividend yield D with $D > 0$. Value a perpetual call on an asset with nonzero dividend yield (I would lead you through this, as I did for the perpetual put in HW 5 problem 3.) [I will not ask you to reproduce the change of variables that transforms Black-Scholes to the linear heat equation, or the reflection principle that led to the valuation formula for a barrier option.]

6. Stochastic interest rates

   - Various representations: discount rates, term rates, etc.
   - The forward rate $F_0(t, T) = B(0, T)/B(0, t)$ and its interpretation.
   - Value of a forward rate agreement.
   - Value of a swap.
   - Binomial interest-rate trees: using a tree to evaluate $B(0, T)$ for various T, to value options, and to hedge options. Finding a tree consistent with market data for $B(0, T)$.

   Sample exam questions: Value a specific forward rate agreement. Value a specific swap, or find the par swap rate (like HW6, problem 1). Given a tree, find the associated discount rates; use it to value an option; use it for hedging (like HW6, problem 5). I would find a way to frame the question so a calculator is not necessary.

7. Caps, floors, and swaptions

   - Black's formula applied to options on zero-coupon bonds; to caps (viewed as sums of caplets) and floors (viewed as sums of floorlets); and to swaptions.
   - Justification of Black's formula by change of numeraire.

   Sample exam questions: For a specific caplet, floorlet, or swaption, explain which version of Black's formula should be used (give an expression for the instrument's value, without actually doing any arithmetic).