**Derivative Securities – Homework 2 – distributed 9/27/04, due 10/11/04**

*Class mailing list: a new mailing list has been set up for this class. To sign up, go to http://cs.nyu.edu/mailman/listinfo/g63_2791_001_fa04.*

Problems 1 and 2 reinforce our discussion of one-period markets (the Section 2 notes). Problems 3-5 reinforce our discussion of multiperiod binomial trees (the Section 3 notes). The classic article on binomial trees, still well worth reading, is J. Cox, S. Ross, and M. Rubinstein, *Option pricing: a simplified approach*, J. Financial Economics 7 (1979) 229-263. It is available online through www.sciencedirect.com (to gain access using a non-NYU computer use the NYU proxy server; for instructions see http://library.nyu.edu/help/proxy.html).

(1) Consider the one-period trinomial model with

- asset 1 = risk-free, interest rate $r > 0$
- asset 2 = risky, initial unit price $s_0$, final unit prices $s_0 d$, $s_0$, $s_0 u$

with $d < 1 < u$. Assume that $d < e^{rT} < u$ so the market admits no arbitrage. You want to buy a call option on the risky asset with strike price $K$. Let's find the largest and smallest prices you should consider paying for it, based on considerations of arbitrage.

(a) Let $\pi_1, \pi_2, \pi_3$ be the risk-neutral probabilities associated to the down, no-change, and up states respectively (these are the $\hat{\pi}$'s of the Section 2 notes). They must satisfy

$$\pi_1 + \pi_2 + \pi_3 = 1, \quad \pi_1 d + \pi_2 + \pi_3 u = e^{rT}, \quad \pi_i \geq 0 \text{ for each } i.$$

These relations restrict $(\pi_1, \pi_2, \pi_3)$ to a line segment. What are its endpoints?

(b) Any contingent claim in this market is described by a vector $f = (f_1, f_2, f_3)$ giving the payoffs if the final-time stock price is $s_0 d$, $s_0$, and $s_0 u$ respectively. Let $V_-(f)$ and $V_+(f)$ be the smallest and largest prices permitted for payoff $f$. Using part (a), give simple formulas for $V_-(f)$ and $V_+(f)$. (Your formulas should involve the min and max of two expressions.) What choice of $f$ corresponds to a call with strike $K$?

(c) The payoff of a call is a *convex* function of the stock price, i..e $F(x) = (s_0 x - K)_+$ is a convex function of $x$. Use Jensen's inequality to show that the min in part (b) is at the endpoint with $\pi_1 = 0$ and the max is at the endpoint with $\pi_2 = 0$. Conclude that for a call with strike $K$ in this 3-period market the smallest and largest prices consistent with the absence of arbitrage are

$$V_- = \frac{ue^{-rT} - 1}{u - 1}(s_0 - K)_+ + \frac{1 - e^{-rT}}{u - 1}(s_0 u - K)_+$$

and

$$V_+ = \frac{ue^{-rT} - 1}{u - d}(s_0 d - K)_+ + \frac{1 - e^{-rT} d}{u - d}(s_0 u - K)_+ \ .$$

(2) Consider the following one-period market with 3 assets and 4 states:

- Asset 1 is a riskless bond, paying no interest.
- Asset 2 is a stock with initial price 1 dollar/share; its possible final prices are $d$ and $u$, with $d < 1 < u$.
- Asset 3 is another stock with initial price 1 dollar/share and possible final prices $d$ and $u$ (same $d$ and $u$).
- To keep the arithmetic simple, let's assume that $u = 1 + \epsilon$ and $d = 1 - \epsilon$ for some $\epsilon > 0$. To avoid confusion, let's number the states: 1 = both stocks go up; 2 = asset 2 goes up, asset 3 goes down; 3 = asset 2 goes down, asset 3 goes up; 4 = both stocks go down.

(a) What system of equations and inequalities characterizes the associated risk-neutral probabilities?

(b) Show that the general solution to (a) is $\pi = (t, \frac{1}{2} - t, \frac{1}{2} - t, t)$ for $0 \leq t \leq 1/2$.

(c) Consider the contingent claim with payoff $f = (f_1, f_2, f_3, f_4)$. Let $V_-(f)$ and $V_+(f)$ be the smallest and largest prices permitted by the absence of arbitrage. Using part (b), give simple formulas for $V_-(f)$ and $V_+(f)$, expressing each as the min or max of two expressions.

(d) Does $f_\alpha \geq 0$ for all $\alpha$ and $V_-(f) = 0$ imply $f = 0$? Explain.

(e) Which $f$'s are replicatable?

Problems 3-5 are from the book by Jarrow and Turnbull. For the binomial trees in these problems please use constant risk-free rate $r$, and $s_{\text{up}} = us_{\text{now}}$and $s_{\text{down}} = ds_{\text{now}}$ with

$$u = e^{[(r-\sigma^2/2)\delta t + \sigma\sqrt{\delta t}]}, \quad d = e^{[(r-\sigma^2/2)\delta t - \sigma\sqrt{\delta t}]}$$

where $r$ is the risk-free rate, $\sigma$ is the volatility of the underlying, and $\delta t$ is the time interval. (We'll discuss the logic behind this choice soon; if you'd like to read ahead, see e.g. Chapter 5 of Jarrow-Turnbull, or my old Section 4 notes.) Convention concerning units: if $r$ and $\sigma$ are given "per year" then $\delta t = 1/2$ for a time period of 6 months, $\delta t = 1/4$ for a time period of 3 months, etc.

(3) A European put option with strike price 45 dollars matures in one year. The underlying asset has volatility 20 percent per annum, and the current spot price is 50 dollars. The risk-free interest rate is 5.60 percent per annum. Divide the one-year interval into two six-month intervals, and use the recombinant tree with $u$ and $d$ as given above.

(a) Show that $u = 1.172832$ and $d = .883891$. Evaluate the risk-neutral probabilities.

(b) Determine the put price by working backward through the tree.

(c) Determine the put price by using the formula which gives it as an average over all final-time payoffs. Of course your answer should be the same as for (b).

(d) Describe the associated trading strategy. In other words, specify how many units of stock and how much debt you should hold at each node after rebalancing.

(4) The current stock price is 100, and the volatility is 30 percent per annum. The risk-free interest rate is 6 percent per annum. Consider a one-year European call option on this stock with strike price 100.

(a) Divide the one-year period into two six-month intervals, and use the recombinant tree with $u$ and $d$ as given above. Calculate the risk-neutral probabilities. Show that the option value is 13.65.

(b) Suppose the market price of the option is 14 7/8. Assuming the market is truly described by the tree of part (a), there must be an arbitrage. Explain in detail (specifying all trades) how you can take advantage of the "incorrect" market price to earn a risk-free profit.

(4) A special kind of one-year put option is written on a stock. The current stock price is 40 and the current strike price is 40. At month 6, if the stock price is below 35 the strike price is lowered to 35; otherwise it remains unchanged. The risk-free interest rate is 5 percent per annum; the volatility of the stock is 35 percent per annum.

(a) Use a 2-period binomial tree with the usual choices of $u$ and $d$ to value the option.

(b) Now use a 4-period binomial tree to value the option.

(c) What is the difficulty with valuing this type of option?