Derivative Securities – Homework 4 – distributed 10/25/04, due 11/08/04

These problems provide some practice with the Black-Scholes PDE (Problem 1) and stochastic differential equations (Problems 2-5).

1) We considered, in HW3, a derivative whose payoff was $s^n(T)$ at maturity, where $s(t)$ has lognormal dynamics with constant volatility σ, and the risk-free rate is r (also constant). We showed there that the derivative has value

$$s^n(t)\exp\left([\tfrac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)\right)$$

at time t. Let's give a different derivation of the same result, using the Black-Scholes PDE.

(a) Substitute $V(s,t) = h(t)s^n$ into the Black-Scholes PDE. What ODE must $h(t)$ solve? What is the appropriate final-time condition?

(b) Verify that $h(t) = \exp\left([\tfrac{1}{2}\sigma^2 n(n-1) + r(n-1)](T-t)\right)$ solves the ODE you found in (a), with the appropriate final-time condition.

(2) Consider the solution of

$$ds = r(t)s\,dt + \sigma(t)s\,dw, \quad s(0) = s_0. \tag{1}$$

where $r(t)$ and $\sigma(t)$ are deterministic functions of time.

(a) Show that $\log s(t)$ is a Gaussian random variable, with mean $\int_0^t [r(s) - \frac{1}{2}\sigma^2(s)]\,ds$ and variance $\int_0^t \sigma^2(s)\,ds$.

(b) Show that $s(T) = s_0 \exp\left([\overline{r} - \tfrac{1}{2}\overline{\sigma}^2]T + \overline{\sigma}\sqrt{T}Z\right)$ where Z is a standard Gaussian,

$$\overline{r} = \frac{1}{T}\int_0^T r(s)\,ds \quad \text{and} \quad \overline{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2(s)\,ds.$$

[Comment: we'll show soon that (1) is the "risk-neutral" stock price process when the risk-free rate and volatility are deterministic functions of t. This problem shows that options can be valued in that setting using the standard Black-Scholes formula, with r replaced by $\overline{r}$ and σ replaced by $\overline{\sigma}$.]

(3) We showed in class using Ito's formula that if $s(t) = s(0)e^{\mu t + \sigma w(t)}$ then $ds = (\mu + \frac{1}{2}\sigma^2)sdt + \sigma s dw$.

(a) Conclude that $E[s(t)] - E[s(0)] = (\mu + \frac{1}{2}\sigma^2)\int_0^t E[s(\tau)]d\tau$, where E denotes expected value.

(b) Conclude that $E[s(t)] = s(0)e^{(\mu + \frac{1}{2}\sigma^2)t}$.

[Comment: taking $t = 1$, this gives a new proof of the lemma, stated at the end of the Section 4 notes, that if X is Gaussian with mean μ and standard deviation σ then $E[e^X] = e^{\mu+\sigma^2/2}$.]

(4) This problem should help you understand Ito's formula. If w is Brownian motion, then Ito's formula tells us that $z = w^2$ satisfies the stochastic differential equation $dz = 2wdw+dt$. Let's see this directly:

(a) Suppose $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$. Show that $w^2(t_{i+1}) - w^2(t_i) = 2w(t_i)(w(t_{i+1}) - w(t_i)) + (w(t_{i+1}) - w(t_i))^2$, whence

$$w^2(b) - w^2(a) = 2\sum_{i=0}^{N-1} w(t_i)(w(t_{i+1}) - w(t_i)) + \sum_{i=0}^{N-1} (w(t_{i+1}) - w(t_i))^2$$

(b) Let's assume for simplicity that $t_{i+1} - t_i = (b-a)/N$. Find the mean and variance of $S = \sum_{i=0}^{N-1}(w(t_{i+1}) - w(t_i))^2$.

(c) Conclude by taking $N \to \infty$ that

$$w^2(b) - w^2(a) = 2\int_a^b w\, dw + (b-a).$$

[Comment: we did parts of this calculation in the notes and in class, but because it's so enlightening I'm asking you to go through it carefully here.]

(5) Here's a cute application of the Ito calculus. Let

$$\beta_k(t) = E[w^k(t)]$$

where $w(t)$ is Brownian motion (with $w(0) = 0$). Show using Ito's formula that for $k = 2, 3, \ldots$,

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\, ds.$$

Deduce that $E[w^4(t)] = 3t^2$. What is $E[w^6(t)]$?

[Comment: the moments of w can also be calculated from its distribution function, since $w(t)$ is Gaussian with mean 0 and variance 1. But the method in this problem is easier, and good practice with Ito's lemma.)