Derivative Securities – Homework 5 – distributed 11/15/04, due 11/29/04

(1) Consider options on an underlying with continuous dividend yield D (assumed constant and positive). Show that the value of a European call with strike K is smaller than $s_0 - K$ when s_0 is sufficiently large. Conclude that it can be optimal to exercise an American call prior to maturity.

(2) Let's value a *perpetual American put* with strike K, written on a non-dividend-paying stock with lognormal dynamics. By definition this instrument never matures, and it can be exercised at any time t yielding payoff $(K - s(t))_+$. Since it never matures, we expect its value to be a function of $s(t)$ alone, i.e. to have the form $V(s(t))$ where

$$\tfrac{1}{2}V_{ss}\sigma^2 s^2 + rsV_s - rV \le 0 \quad \text{and} \quad V(s) \ge (K-s)_+$$

for all s. Moreover we expect that there is an exercise threshold s_* such that

$$\tfrac{1}{2}V_{ss}\sigma^2 s^2 + rsV_s - rV = 0 \quad \text{for } s \ge s_*,$$

and

$$V(s) = (K-s)_+ \quad \text{for } s \le s_*,$$

and we expect that V and V_s are continuous at s_*. Finally we expect that $V(s) \to 0$ as $s \to \infty$.

(a) Show that if $k = 2r/\sigma^2$ then $f(s) = As + Bs^{-k}$ solves the PDE $\frac{1}{2}f''\sigma^2 s^2 + rsf' - rf = 0$ for any choice of the constants A and B. (If you know some PDE then you'll recognize that this is the most general possible solution.)

(b) Show that to have $f(s_*) = (K - s_*)$ and $f(\infty) = 0$ we must set $A = 0$ and $B = s_*^k(K - s_*)$.

(c) Show that the choice $s_* = \frac{k}{1+k}K$ gives $f'(s_*) = -1$, and that the resulting function $f(s)$ has all the properties listed above for $V(s)$. This is the desired function $V(s)$ which values the perpetual American put.

(3) [Jarrow-Turnbull, chapter 7, problem 2]. An American put option with a maturity of one year and a strike price of 60 is written on a non-dividend-paying stock. Assume the current stock price s_0 is 60, the volatility σ is 35 percent per year, and the risk-free rate r is 6 percent per year. To keep things simple, let's use a two-period binomial tree to value the option.

(a) Construct an appropriate two-period recombining price tree using $u = \exp[(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}]$ and $d = \exp[(r - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}]$. Notice that when δt is 6 months, $u = 1.2800$, $d = .7803$, and the risk-neutral probability of the up state is $q = .5007$. (You may round this to $q = .5$.) Value the option by working backward through the tree. Don't forget to check whether early exercise is optimal.

(b) Describe the replicating portfolio at each node. Verify that the associated trading strategy is self-financing, and that it replicates the payoff.

(4) [like Jarrow-Turnbull, chapter 11, problem 5]. Consider a call option written on Euros with a maturity of one year. The spot exchange rate is 1.30 dollars/Euro, its volatility is 12 percent per year, and the strike price is 1.1818. The contract size is $250,000$ Euros. To keep things simple, let's use a two-period binomial tree to value the option. Assume that if you invest one dollar for six months at the risk-free rate it will be worth 1.0151 dollars, and if you invest one Euro for six months at the risk-free rate it will be worth 1.0305 Euros.

(a) Construct an appropriate two-period recombining price tree using $u = \exp[(r - D - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}] = 1.068436$ and $d = \exp[(r - D - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}] = .901667$.

(b) Explain briefly how we should use this tree to value the option. How does the Euro interest rate affect the calculation?

(c) If the option is European (exercisable only at maturity) what is its value? (You may round off the risk-neutral probability to $q = .5$).

(d) If the option is American (exercisable at six months or at maturity) what is its value?

(e) Describe the hedging strategy at each node. (The hedge portfolio consists of a Euro bond holding and a dollar bond holding. Notice that each bond earns interest at the appropriate risk-free rate. I am asking for the *dollar-investor's* hedge portfolio; equivalently, I'm asking for the trading strategy a dollar investor can use to replicate the option.)

5) The Section 9 notes discuss use of Black's formula to price options on futures. Focusing for simplicity on calls: if the futures price F_t is lognormal with volatility σ_F then the value of a European call with maturity T (whose payoff is $(F_T - K)_+$ at time T)is $e^{-rT}[F_0 N(d_1) - KN(d_2)]$, where $d_1 = [\log(F_0/K) + \frac{1}{2}\sigma^2 T]/\sigma\sqrt{T}$ and $d_2 = [\log(F_0/K) - \frac{1}{2}\sigma^2 T]/\sigma\sqrt{T}$.

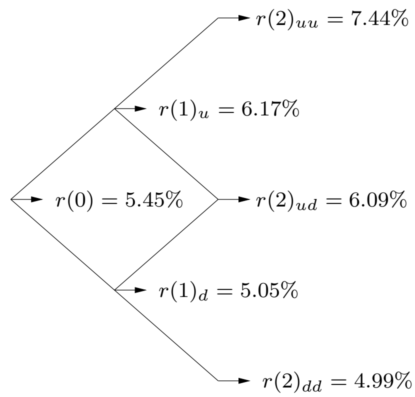
(a) Suppose the underlying is a non-dividend paying stock with volatility σ, and the futures prices under consideration are for delivery at time T_F. What value do you use for σ_F in Black's formula?

(b) This call can be hedged by continuously trading futures contracts (with delivery time T_F) and the risk-free asset. Describe the appropriate trading strategy.

(c) The call can alternatively be hedged by continuously trading the underlying and the risk-free asset. Describe the appropriate trading strategy.

6) [Jarrow & Turnbull, Chapter 15, problem 4.] Consider the binomial tree of interest rates shown in the figure (each time interval is one year, and the rates shown are per annum with continuous compounding). Assume the risk-neutral probabilities are 1/2 for each branch.

(a) Find the values of $B(0,1)$, $B(0,2)$, and $B(0,3)$.



(b) Consider the following European call option written on a one year Treasury bill: its maturity is $T = 2$, and its strike is 0.945, so the payoff at time 2 is $(B(2,3) - 0.945)_+$. Find the value of this option at time 0.

(c) Suppose you wish to hedge this option using two-year and three-year treasury bills. Find the hedge portfolio at time 0. [Hint: to find the hedge portfolio at a given node, solve an appropriate system of two linear equations in two unknowns.]