

Derivative Securities – Fall 2004 – Semester Review

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1. One-period market models

- Arbitrage-based pricing: value of a forward contract on a non-dividend-paying stock; value of a forward contract for foreign currency; put-call parity; etc.
- Binomial market: value any contingent claim by finding a hedge portfolio. Expression for the value as the discounted risk-neutral expectation. The risk-neutral probability is uniquely determined by the condition that it give the right value for a forward.
- Trinomial and more general markets: we can give upper and lower bounds for the value of a contingent claim by solving a pair of linear programming problems. The dual problem is an optimization over risk-neutral probabilities. A contingent claim is replicatable exactly if the upper and lower bounds coincide.

Sample exam questions: value a forward contract by identifying a portfolio of known value with the same payoff at time T . Or take advantage of a mispriced forward – how much can you gain through this arbitrage?.

2. Multiperiod binomial trees

- Valuing contingent claims: working backward in the tree.
- Hedging: dynamic replication of a contingent claim.
- The valuation formula: for a European option, value = discounted risk-neutral expectation of payoff at maturity.
- American options: check at each node for the possibility of early exercise.
- Continuous dividend yield, foreign currency, options on futures: similar framework, but the formula for the risk-neutral probability is different
- Passage to the continuous-time limit: application of the central limit theorem. (Don't confuse the subjective and risk-neutral processes.)
- Futures prices. (Note: the futures price is a martingale under the risk-neutral probability; it is not the price of a tradeable.)

Sample exam questions: Value a contingent claim by working backward in a tree. Specify the hedging strategy, including rebalancing. What if the option is American, i.e. permits early exercise? When considering an option on foreign currency, what is the proper choice of the risk-neutral probability and why? Consider an N -step multiplicative tree with $u = \exp(\mu\delta t + \sigma\sqrt{\delta t})$ and $d = \exp(\mu\delta t - \sigma\sqrt{\delta t})$; if the probability of going up is p and the probability of going down is $1 - p$, find the mean and variance of $\log s(N\delta t)$.

3. Derivation and use of the Black-Scholes formulas

- I'll give you the formula for $E[e^{aX}$ restricted to $X \geq k]$ if you need it. But you'll need to know how to choose the mean and variance of X , what to use for a , etc. for a specific application.
- Option values: deriving the BS formula for valuing a put or a call; interpretations of the terms; analogous formulas for other options such as a powered call.
- Hedging: deriving the formulas for Delta, Vega, etc.
- Qualitative properties, for puts and calls. Early exercise can be optimal for an American put.
- Continuous dividend yield, options on futures: Black's formula

Sample exam questions: for the risk-neutral price process, find the probability that its value at time T is greater than K . Derive a formula for the value or the Delta of a particular option (like HW3 problem 6, valuing an option with payoff s_T^n ; or HW4 problem 1, concerning an option with payoff $(s_T - K)_+^2$).

4. The basic continuous-time theory

- Stochastic differential equations. The lognormal stock process as a special case.
- Applications of Ito's formula. Consequences of the fact that dw integrals have mean value 0 (are martingales).
- Derivation of the Black-Scholes PDE based on hedging and rebalancing.
- Interpretation of the Black-Scholes PDE: $e^{-rt}V(s(t), t)$ is a martingale for the risk-neutral price process, which solves $ds = rsdt + \sigma sdw$.
- Equivalence of pricing based on the solution formula (value = discounted expected payoff, using the risk-neutral probabilities) and based on solving the Black-Scholes PDE (value = $V(s_0, 0)$ where V solves the PDE with the option payoff as final-time at $t = T$.)

Sample exam questions: show that if $ds = rsdt + \sigma sdw$ then $\log s(t)$ is Gaussian with mean $r - (1/2)\sigma^2$ and variance $\sigma^2 t$. Show that the solution of $ds = rsdt + \sigma sdw$ has the property that $s(t)e^{-rt}$ is a martingale, i.e. $E[s(t)e^{-rt}]$ is independent of time; show further that if $V(s, t)$ solves the Black-Scholes differential equation then $V(s(t), t)e^{-rt}$ is a martingale, i.e. $E[V(s(t), t)e^{-rt}]$ is independent of time; use this to connect our two methods for finding the value of an option (by solving the Black-Scholes PDE, and by evaluating the discounted expected payoff using the risk-neutral process.) Derive the Black-Scholes PDE by considering a suitable hedging strategy.

5. Further continuous-time theory

- Equivalence of the Black-Scholes PDE and the linear heat equation
- American options

Sample exam questions: show that early exercise is never optimal, for an American call on a non-dividend-paying stock. Show that it can be optimal for an American put. Show that it can be optimal on an American call, if the underlying has continuous

dividend yield D with $D > r$. Value a perpetual call on an asset with nonzero dividend yield [I would not ask you to reproduce from memory the change of variables that transforms Black-Scholes to the linear heat equation.]

6. Stochastic interest rates

- Various representations: discount rates, term rates, etc.
- The forward rate $F_0(t, T) = B(0, T)/B(0, t)$ and its interpretation.
- Value of a forward rate agreement.
- Value of a swap.
- Binomial interest-rate trees: using a tree to evaluate $B(0, T)$ for various T , to value options, and to hedge options.

Sample exam questions: Value a specific forward rate agreement. Value a specific swap, or find the par swap rate. Given a tree, find the associated discount rates; use it to value an option; hedge the option, e.g. by a suitable portfolio of bonds.

7. Caps, floors, and swaptions

- Black's formula applied to options on zero-coupon bonds; to caps (viewed as sums of caplets) and floors (viewed as sums of floorlets); and to swaptions.
- Justification of Black's formula by change of numeraire.

Sample exam questions: For a specific caplet, floorlet, or swaption, explain which version of Black's formula should be used (give an expression for the instrument's value, without actually doing any arithmetic). Explain, on a tree, why if f and g are tradeables then there's a measure that makes f/g a martingale.

8. Credit risk

- Default probabilities and their relationship with defaultable bond prices.
- Credit default swaps; the CDS spread.
- Merton's framework for relating equity prices with default probabilities.

Sample exam equations: extract default probabilities from bond prices; evaluate a CDS spread. [New material covered on 12/13, like the last bullet above, will not be on the exam.]