Derivative Securities, Fall 2012 – Homework 3. Distributed 10/10. This HW set is long, so I'm allowing 3 weeks: it is due by classtime on 10/31. *Typos in pbm 3 (ambiguity in the payoff of a squared call) and pbm 6 (inconsistent notation for the dividend rate) have been corrected here.*

Problem 1 provides practice with lognormal statistics. Problems 2-6 explore the consequences of our formula for the value of an option as the discounted risk-neutral expected payoff. Problem 7 makes sure you have access to a numerical tool for playing with the Black-Scholes formula and the associated "Greeks." Problem 8 reinforces the notion of "implied volatility."

Convention: when we say a process X_t is "lognormal with drift μ and volatility σ" we mean $\ln X_t - \ln X_s$ is Gaussian with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$ for all $s < t$; here μ and σ are constant. In this problem set, X_t will be either a stock price or a forward price. The interest rate is always r, assumed constant.

(1) Suppose a stock price s_t is lognormal with drift μ and volatility σ, and the stock price now is s_0.

(a) Give a 95% confidence interval for s_T, using the fact that with 95% confidence, a Gaussian random variable lies within 1.96 standard deviations of its mean.

(b) Give the mean and variance of s_T.

(c) Give a formula for the likelihood that a call with strike K and maturity T will be in-the-money at maturity.

(d) If the drift is 16% per annum and the volatility is 30% per annum, what do (a) and (b) tell you about tomorrow's closing price in terms of today's closing price?

(e) What is the probability that $s_T > E[s_T]$? (Note: the answer is not 1/2.)

(2) Suppose a stock price s_t is lognormal. Consider a derivative with payoff s_T^n at maturity. (To be sure there's no confusion: the payoff is s_T raised to the power n.) Show that its value at time t is

$$s_t^n e^{[\frac{1}{2}\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free rate and σ is the volatility. (Hint: the option's value is $e^{-rT}E_{\mathrm{RN}}[\text{payoff}]$.)

(3) Consider a squared call with strike K and maturity T, i.e. an option whose payoff at maturity is $[(s_T - K)_+]^2$.

(a) Give a formula for the value of the squared call at time 0, analogous to the standard formula $s_0 N(d_1) - Ke^{-rT}N(d_2)$ for an ordinary call.

(b) Evaluate its hedge ratio (its "Delta") by differentiating under the integral, then evaluating the resulting expression.

(Hint: For part (a) use the fact that $(e^x - K)^2 = e^{2x} - 2Ke^x + K^2$. Concerning part (b): one could of course differentiate the answer to (a) to find Delta, but that's the hard way.)

(4) Consider a "cash-or-nothing" option with strike price K, i.e. an option whose payoff at maturity is

$$f(s_T) = \begin{cases} 1 & \text{if } s_T \geq K \\ 0 & \text{if } s_T < K \end{cases}$$

It can be interpreted as a bet that the stock will be worth at least K at time T.

(a) Give a formula for its value at time t, in terms of the spot price s_t.

(b) Give a formula for its Delta (i.e. its hedge ratio). How does the Delta behave as t gets close to T?

(c) Why is it difficult, in practice, to hedge such an instrument?

In view of (c) it is not entirely clear that the Black-Scholes valuation formula is valid for such an option. What do you think?

(5) Derive all the formulas given in the Section 5 notes for the "Greeks" of calls and puts on a forward price. Make use of the following hints:

(a) In each case, derive the formula for the call option first; then derive the formula for the put option by using put-call parity.

(b) Use the fact that $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, which follows by the Fundamental Theorem of Calculus from the definition $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$.

(c) To derive the formulas for the delta and vega of a call, start by showing that $F_0 N'(d_1) = K N'(d_2)$.

(6) Suppose a stock price s_t is lognormal, with continuous dividend yield q. Hull explains in Section 16.3 why calls and puts on s_t can be priced by replacing s_0 with $s_0 e^{-qT}$ in the standard Black-Scholes formula; the resulting prices are Hull's eqns (16.4)-(16.5). Give an alternative derivation of those pricing formulas, using the expressions given in the Section 5 notes for the value of a call or put on a lognormal forward price.

(7) Suppose r is 5 percent per annum and σ is 20 percent per annum. Let's consider standard put and call options on a forward price F_t with strike price $K = 50$. Do this problem using the Black-Scholes formulas (not a binomial tree).

(a) Suppose the forward price is $F_0 = 50$ and the maturity is one year. Find the value, delta, vega, and gamma of the put. Same request for the call. Do the same computation for $F_0 = 40$ and $F_0 = 60$.

(b) Graph the value of a European call as a function of the forward price F_0, for maturities of 1, 2 and 3 years. Display all the graphs on a single set of axes, and comment on the trends they reveal.

(c) Same as (b) but for a European put.

[Comment: Use whatever means (matlab, mathematica, spreadsheet) is most convenient, but say briefly what you used. One point of this problem is to visualize the behavior of the pricing formulas. Another is to be sure you have a convenient tool for exploring further on your own.]

(8) Let's estimate the implied volatility of a 3 year European-style call option on a forward price, if the forward price is presently 100, the strike is 90, the risk-free rate is 4% per annum, and the option price is 21.05. Start with a volatility of 20% and determine the option price and vega predicted by the Black-Scholes theory. Do three iterations of Newton's method, guessing a new volatility based on the previous call price and vega, then re-estimating call price and vega.