A compactness result in the gradient theory of phase transitions

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Abstract

We examine the singularly perturbed variational problem $E_\epsilon(\psi) = \int \epsilon^{-1}(1-|\nabla\psi|^2)^2 + \epsilon|\nabla\nabla\psi|^2$ in the plane. As $\epsilon \to 0$ this functional favors $|\nabla\psi| = 1$ and penalizes singularities where $|\nabla\nabla\psi|$ concentrates. Our main result is a compactness theorem: if $E_\epsilon(\psi_\epsilon)$ is uniformly bounded then $\nabla\psi_\epsilon$ is compact in L^2. Thus, in the limit $\epsilon \to 0$ ψ solves the eikonal equation $|\nabla\psi| = 1$ almost everywhere. Our analysis uses "entropy relations" and the "div-curl lemma," adopting Tartar's approach to the interaction of linear differential equations and nonlinear algebraic relations.

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1 Motivation, statement of the result and idea of the proof

We consider the singularly perturbed functional

$$E_\epsilon(\psi) \;=\; \epsilon \int_\Omega |\nabla\nabla\psi|^2 + \frac{1}{\epsilon}\int_\Omega (1-|\nabla\psi|^2)^2 \tag{1}$$

as $\epsilon \downarrow 0$. It arises as a model problem in connection with several physical applications, including smectic liquid crystals (see Aviles & Giga [2]), thin film blisters (see Ortiz & Gioia [8, 17]), and convective pattern formation (see Ercolani *et. al* [7]). Physically (1) can be viewed as a simple Landau theory, in which the order parameter is a curl-free vector field $\nabla\psi$ which prefers to be of norm 1.
The functional analysis of (1) is still poorly understood, despite considerable attention. A natural goal is to find the "asymptotic energy" as $\epsilon \downarrow 0$, represented by the Γ–limit of E_ϵ (see for instance [6]). A formula for this asymptotic energy was conjectured by Aviles and Giga [2]: it minimizes a certain "fold energy," as ψ ranges over almost-everywhere solutions of the eikonal equation $|\nabla\psi| = 1$. To confirm their conjecture, one needs to show (informally speaking) that:

(a) solutions of the eikonal equation are the appropriate admissible set;

(b) the proposed formula for the fold energy is correct, i.e. energetically optimal folds are "locally one-dimensional;"

(c) the asymptotic energy lives only on the folds, i.e. lower-dimensional singularities carry no energy.

All the analysis to date has been restricted to the case when space is two-dimensional. Point (a) is demonstrated in the present paper. Point (b) is substantially confirmed by the work of Jin & Kohn [9, 10] and Aviles & Giga [3]. Point (c) is basically open. After this work was done but before it was submitted for publication we learned of related progress by Ambrosio *et al.* [1]. They also demonstrate (a), using a method entirely different from ours, and they show by example that the admissible ψ's can be unexpectedly complex.

Our functional (1) is an obvious generalization to gradient fields of the scalar problem considered by Modica & Mortola (in [13], [14] and [12])

$$\tilde{E}_\epsilon(u) \;=\; \epsilon \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon}\int_\Omega (1-u^2)^2. \tag{2}$$

Let us briefly review the compactness result associated with (2). The precise statement is: if, for a $\{u_\epsilon\}_{\epsilon\downarrow 0}$, the energies $\{\tilde{E}_\epsilon(u_\epsilon)\}_{\epsilon\downarrow 0}$ are uniformly bounded, then $\{u_\epsilon\}$ is relatively compact in $L^2(\Omega)$. The essence of the argument is this estimate for $v_\epsilon = u_\epsilon\,(1-\frac{1}{3}\,u_\epsilon^2)$

$$\begin{aligned}\int_\Omega |\nabla v_\epsilon| \;&=\; \int_\Omega |(1-u_\epsilon^2)\nabla u_\epsilon| \\ &\leq\; \left(\int_\Omega |\nabla u_\epsilon|^2\right)^{\frac{1}{2}} \left(\int_\Omega (1-u_\epsilon^2)^2\right)^{\frac{1}{2}} \\ &\leq\; \frac{\epsilon}{2}\int_\Omega |\nabla u_\epsilon|^2 + \frac{1}{2\,\epsilon}\int_\Omega (1-u_\epsilon^2)^2 \;=\; E_\epsilon(u_\epsilon). \end{aligned} \tag{3}$$

The estimate implies the boundedness of $\{\nabla v_\epsilon\}_{\epsilon\downarrow 0}$ in $L^1(\Omega)$, which provides sufficient compactness. It is obvious that the above argument does not generalize to (1): There is no equivalent to (3), since there is no transformation Φ such that $\mathrm{D}[\Phi(\nabla\psi_\epsilon)] = (1-|\nabla\psi_\epsilon|^2)\,\mathrm{D}^2\psi_\epsilon$. The difference may also be seen as follows: In case of (2), the favored values of u form a discrete set $\{-1,1\}$. In case of (1), the favored values of $f=\nabla\psi$ form a continuum $\{|z|^2=1\}$. Hence in case of (1), the additional information that $\nabla\times f = 0$ is essential for compactness. We will have to investigate the combined effect of the linear differential equation $\nabla\times f=0$ and the nonlinear relation $|f|^2=1$.

Proposition 1 *Let $\Omega\subset I\!R^2$ open and bounded. Let the sequences $\{\epsilon_\nu\}_{\nu\uparrow\infty}\subset(0,\infty)$ and $\{\psi_\nu\}_{\nu\uparrow\infty}\subset H^2(\Omega)$ be such that*

$$\epsilon_\nu \overset{\nu\uparrow\infty}{\longrightarrow} 0 \quad \textit{and} \quad \{E_{\epsilon_\nu}(\psi_\nu)\}_{\nu\uparrow\infty} \quad \textit{is bounded.}$$

Then

$$\{\nabla\psi_\nu\}_{\nu\uparrow\infty}\subset L^2(\Omega) \quad \textit{is relatively compact.}$$

Actually we prove a bit more than Proposition 1. To state the stronger result, we prefer to work with the divergence-free vector fields $m_\nu = \mathrm{R}\nabla\psi_\nu$, where R denotes rotation by $\frac{\pi}{2}$, that is $\mathrm{R}\binom{z_1}{z_2} = \binom{-z_2}{z_1}$. This shift of perspective entails no loss of generality (our method seems intrinsically limited to two space dimensions). Moreover it highlights the analogy between (1) and the micromagnetic energy of an isotropic thin film, where m is only approximately divergence-free, but $|m| = 1$ exactly. In truth, we first found the arguments behind Proposition 2 while exploring the micromagnetics of thin films. This paper focuses on (1) instead of micromagnetics, because that is the more familiar and widely-studied problem. Our stronger result is:

Proposition 2 *Let $\Omega \subset \mathbb{R}^2$ open and bounded. Let the sequence $\{m_\nu\}_{\nu\uparrow\infty} \subset H^1(\Omega)$ be such that*

$$\nabla \cdot m_\nu \;=\; 0 \quad \textit{a. e. in}\, \Omega, \tag{4}$$

$$\|1 - |m_\nu|^2\|_{L^2(\Omega)} \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; 0, \tag{5}$$

$$\Big\{\|\nabla m_\nu\|_{L^2(\Omega)}\, \|1 - |m_\nu|^2\|_{L^2(\Omega)}\Big\}_{\nu\uparrow\infty} \quad \textit{is bounded.} \tag{6}$$

Then

$$\{m_\nu\}_{\nu\uparrow\infty} \subset L^2(\Omega) \quad \textit{is relatively compact.} \tag{7}$$

The fact that this is a non trivial issue becomes apparent by the following argument: Assume that (7) is true. Then there exists an $m \in L^2(\Omega)$ such that for a subsequence

$$m_\nu \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; m \quad \text{in } L^2(\Omega).$$

Property (4) is conserved in the limit in a weak sense:

$$\nabla \cdot m \;=\; 0 \quad \text{in a distributional sense on } \Omega, \tag{8}$$

whereas (5) sharpens into

$$|m|^2 \;=\; 1 \quad \text{a. e. in } \Omega. \tag{9}$$

On the level of $L^2(\Omega)$–functions, the combination of the linear partial differential equation (8) and the nonlinear relation (9) is not enough to ensure compactness in $L^2(\Omega)$. On the level of differentiable functions, it is very rigid. (This can be easily seen by going back to the original description $m = \mathrm{R}\nabla\psi$ in which (8) is automatically fulfilled and (9) turns into the eikonal equation

$|\nabla\psi|^2 = 1$.) Hence in our compactness proof, we will have to combine the linear partial differential equation (4), the increasing penalization of $|m|^2 \neq 1$ through (5), and the (fading) control of $\mathrm{D}m$ through (6).

Let us sketch the basic idea of the proof of Proposition 2. For this, we reconsider an m which satisfies both the linear partial differential equation (8) and the nonlinear relation (9). Because of (9), we can write $m = \binom{\cos\theta}{\sin\theta}$ with a function θ so that (8) turns into

$$\partial_1(\cos\theta) + \partial_2(\sin\theta) \;=\; 0. \tag{10}$$

It is enlightening to think of (10) as a scalar conservation law for the quantity $s \simeq \cos\theta$ which depends on time $t \simeq x_1$ and a single spatial variable $y \simeq x_2$:

$$\partial_t s + \partial_y f(s) \;=\; 0. \tag{11}$$

As a scalar conservation law (11), (10) would be highly nonlinear. As can be seen by the method of characteristics, (11) with a nonlinear flux function f generically does not admit differentiable solutions to the Cauchy problem. On the other hand, there generically are infinitely many distributional solutions to the Cauchy problem. The physically motivated notion of entropy solution has been introduced; the Cauchy problem is well–posed in this framework, see for instance [11].

What is the notion of an entropy solution? If the pair of nonlinear functions (η, q) satisfies $q' = \eta' f'$ (a so–called entropy entropy–flux pair) and if s is a differentiable solution of (11), then

$$\partial_t \eta(s) + \partial_y q(s) \;=\; 0. \tag{12}$$

But if f is nonlinear and s is only a distributional solution of (11), then (12) is generically not satisfied — even in a distributional sense. An entropy solution s of (11) is defined as a distributional solution of (11) with the property that

$$\partial_t \eta(s) + \partial_y q(s) \;\leq\; 0$$

in a distributional sense for all entropy entropy–flux pairs (η, q) such that η is convex. Even if η is not convex, we have for an entropy solution that

$$\partial_t \eta(s) + \partial_y q(s) \quad \text{is a measure.}$$

By a lemma of Murat [16], this implies that if $\{s_\nu\}_{\nu\uparrow\infty}$ is a sequence of uniformly bounded entropy solutions, then

$$\partial_t\eta(s_\nu)+\partial_y q(s_\nu)\quad\text{is compact in } H^{-1}.$$

The latter allows for a judicious application of Murat and Tartar's div–curl lemma (a special case of compensated compactness, see [15] and [18]). Tartar uses this to derive restrictions on the Young measure generated by $\{m_\nu\}_{\nu\uparrow\infty}$ [18]. This allows him to conclude that the set of uniformly bounded entropy solutions is compact, provided f is sufficiently nonlinear. In fact, the scope of his method is more general: It explores how the combination of linear partial differential equations (like (8)) and nonlinear relations (like (9)) restricts and may rule out oscillations. The general tool–box Tartar assembled is perfectly suited for our situation.

In the first part of Section 2 (Lemma 1 and Lemma 2), we will identify all (nonlinear) functions Φ of m with the property that $\Phi(m)$ satisfies a certain linear partial differential equation, provided m satisfies the linear partial differential equation (8) and the nonlinear relation (9). More precisely, we will identify all Φ such that

$$\text{if } m \text{ is differentiable with}\quad \nabla\cdot m\;=\;0\quad\text{and}\quad |m|^2\;=\;1,$$
$$\text{then}\quad \nabla\cdot[\Phi(m)]\;=\;0.$$

This is in the spirit of Tartar and mimics the tool of entropy and entropy–flux pairs (η,q). In the second part of Section 2 (Lemma 5), we will show that the class of entropies is rich enough for our purposes. This doesn't come as a surprise, since the set of all entropy and entropy–flux pairs (η,q) is rich enough for a scalar conservation law in one space dimension (11). In the first part of Section 3, we will show that the control expressed in (6) is strong enough to ensure that for our sequence $\{m_\nu\}_{\nu\uparrow\infty}$

$$\nabla\cdot[\Phi(m_\nu)]\quad\text{is compact in } H^{-1}\quad\text{for above } \Phi's.$$

Then, in the second part of Section 3, we will apply Tartar's program.

2 Entropies

Definition 1 *A $\Phi\in C_0^\infty(I\!R^2)^2$ is called entropy if*

$$z\cdot \mathrm{D}\Phi(z)\mathrm{R}z\;=\;0\quad\textit{for all } z\quad\textit{and}\quad \Phi(0)\;=\;0,\;\mathrm{D}\Phi(0)\;=\;0,\tag{13}$$

where $\mathrm{D}\Phi_{i,j} = \frac{\partial \Phi_i}{\partial x_j}$ *denotes the Jacobian of* Φ *and* R *the rotation by* $\frac{\pi}{2}$*, that is* $\mathrm{R}\binom{z_1}{z_2} = \binom{-z_2}{z_1}$.

Lemma 1 *Let* $\Phi \in C_0^\infty(I\!R^2)^2$ *be an entropy. Then there exists a* $\Psi \in C_0^\infty(I\!R^2)^2$ *such that*

$$\mathrm{D}\Phi(z) + 2\,\Psi(z) \otimes z \quad \textit{is isotropic for all } z. \tag{14}$$

Proof of Lemma 1. Componentwise, (14) is equivalent to the three equations

$$\Phi_{1,1}(z) + 2\,\Psi_1(z)\,z_1 \;=\; \Phi_{2,2}(z) + 2\,\Psi_2(z)\,z_2 \quad \text{and} \tag{15}$$

$$\Phi_{1,2}(z) + 2\,\Psi_1(z)\,z_2 \;=\; 0, \qquad \Phi_{2,1}(z) + 2\,\Psi_2(z)\,z_1 \;=\; 0. \tag{16}$$

By continuity, (15) is equivalent to (15) multiplied with $z_1\,z_2$, that is

$$z_1\,z_2\,\Phi_{1,1}(z) + 2\,z_1^2\,z_2\,\Psi_1(z) \;=\; z_1\,z_2\,\Phi_{2,2}(z) + 2\,z_1\,z_2^2\,\Psi_2(z).$$

Hence the conjunction of (15) and (16) is equivalent to the conjunction of

$$z_1\,z_2\,\Phi_{1,1}(z) - z_1^2\,\Phi_{1,2}(z) \;=\; z_1\,z_2\,\Phi_{2,2}(z) - z_2^2\,\Phi_{2,1}(z) \tag{17}$$

and (16). But (17) is just (13) written in a componentwise fashion and (16) can be satisfied by choosing

$$\Psi_1(z) \;=\; -\frac{1}{2\,z_2}\,\Phi_{1,2}(z) \quad \text{and} \quad \Psi_2(z) \;=\; -\frac{1}{2\,z_1}\,\Phi_{2,1}(z).$$

We observe that by definition we have $\mathrm{D}\Phi(0) = 0$, which ensures $\Psi \in C_0^\infty(I\!R^2)^2$.

Lemma 2 *Let* $\Phi \in C_0^\infty(I\!R^2)^2$ *and* $\Psi \in C_0^\infty(I\!R^2)^2$ *satisfy (14). Let* $m \in H^1(\Omega)^2$ *satisfy*

$$\nabla \cdot m \;=\; 0 \quad \textit{a. e. in } \Omega.$$

Then

$$\nabla \cdot [\Phi(m)] \;=\; \Psi(m) \cdot \nabla(1 - |m|^2) \quad \textit{a. e. in } \Omega.$$

Proof of Lemma 2. We have a. e. in Ω

$$\nabla \cdot [\Phi(m)] \;=\; \mathrm{tr}\, \mathrm{D}[\Phi(m)] \;=\; \mathrm{tr}(\mathrm{D}\Phi(m)\, \mathrm{D}m) \;=\; \mathrm{tr}(\mathrm{D}m\, \mathrm{D}\Phi(m))$$

and

$$\Psi(m) \cdot \nabla(1-|m|^2) \;=\; -2\, m \cdot \mathrm{D}m\, \Psi(m) \;=\; -\mathrm{tr}\left(\mathrm{D}m\, (2\, \Psi(m) \otimes m)\right),$$

so that

$$\nabla \cdot [\Phi(m)] - \Psi(m) \cdot \nabla(1-|m|^2) \;=\; \mathrm{tr}\left(\mathrm{D}m\, (\mathrm{D}\Phi(m) + 2\, \Psi(m) \otimes m)\right).$$

By assumption for a. e. $x \in \Omega$,

$$\begin{array}{rl} \mathrm{tr}(\mathrm{D}m(x)) \;=\; (\nabla \cdot m)(x) \;=\; 0 & , \\ \mathrm{D}\Phi(m(x)) + 2\, \Psi(m(x)) \otimes m(x) & \text{is isotropic} \end{array}$$

and therefore

$$\mathrm{tr}\left(\mathrm{D}m(x)\, (\mathrm{D}\Phi(m(x)) + 2\, \Psi(m(x)) \otimes m(x))\right) \;=\; 0.$$

Lemma 3 *There is a one–to–one correspondence between entropies $\Phi \in C_0^\infty(I\!R^2)^2$ and functions $\varphi \in C_0^\infty(I\!R^2)$ with $\varphi(0) = 0$ via*

$$\Phi(z) \;=\; \varphi(z)\, z + \nabla\varphi(z) \cdot \mathrm{R}z\, \mathrm{R}z. \tag{18}$$

Proof of Lemma 3. Let $\varphi \in C_0^\infty(I\!R^2)$ with $\varphi(0) = 0$ be given and Φ defined via (18). Obviously, $\Phi(0) = 0$. We have

$$\begin{array}{rcl} \mathrm{D}\Phi(z) & = & z \otimes \nabla\varphi(z) + \varphi(z)\, \mathrm{id} \\ & + & \mathrm{R}z \otimes (\mathrm{D}^2\varphi(z)\, \mathrm{R}z - \mathrm{R}\, \nabla\varphi(z)) + \nabla\varphi(z) \cdot \mathrm{R}z\, \mathrm{R} \end{array}$$

and therefore $\mathrm{D}\Phi(0) = 0$ and

$$z \cdot \mathrm{D}\Phi(z)\, \mathrm{R}z \;=\; |z|^2\, \nabla\varphi(z) \cdot \mathrm{R}z + \nabla\varphi(z) \cdot \mathrm{R}z\, z \cdot \mathrm{R}\mathrm{R}z \;=\; 0.$$

On the other hand, let $\Phi \in C_0^\infty(I\!R^2)^2$ be an entropy. Since $\Phi(0) = 0$ and $\mathrm{D}\Phi(0) = 0$,

$$|z|^2\, \varphi(z) \;=\; \Phi(z) \cdot z \tag{19}$$

defines a $\varphi \in C_0^\infty(I\!R^2)$ with $\varphi(0)=0$. Differentiating the identity (19) in the direction $\mathrm{R}z$ yields

$$|z|^2\,\nabla\varphi(z)\cdot\mathrm{R}z \;=\; z\cdot\mathrm{D}\Phi(z)\cdot\mathrm{R}z+\Phi(z)\cdot\mathrm{R}z \;\stackrel{(13)}{=}\; \Phi(z)\cdot\mathrm{R}z. \qquad (20)$$

Hence

$$\begin{aligned}|z|^2\,\Phi(z) &= \Phi(z)\cdot z\,z+\Phi(z)\cdot\mathrm{R}z\,\mathrm{R}z\\ &\stackrel{(19,20)}{=} |z|^2\,\varphi(z)\,z+|z|^2\,\nabla\varphi(z)\cdot\mathrm{R}z\,\mathrm{R}z\\ &= |z|^2\left(\varphi(z)\,z+\nabla\varphi(z)\cdot\mathrm{R}z\,\mathrm{R}z\right).\end{aligned}$$

By continuity, this implies (18).

Lemma 4 *Fix an $e\in S^1$. Then*

$$\Phi(z) \;=\; \left\{\begin{array}{cc} |z|^2\,e & \textit{for } z\cdot e>0\\ 0 & \textit{for } z\cdot e\le 0\end{array}\right\} \qquad (21)$$

is a generalized entropy in the sense that there exists a sequence $\{\Phi_\nu\}_{\nu\uparrow\infty}$ of entropies in $C_0^\infty(I\!R^2)^2$ s. t.

$$\{\Phi_\nu(z)\}_{\nu\uparrow\infty} \quad \textit{is bounded uniformly for bounded } z, \qquad (22)$$

$$\Phi_\nu(z) \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \Phi(z) \quad \textit{for all } z. \qquad (23)$$

Proof of Lemma 4. Consider the function φ

$$\varphi(z) \;=\; \left\{\begin{array}{cc} z\cdot e & \text{for } z\cdot e>0\\ 0 & \text{for } z\cdot e\le 0\end{array}\right\}$$

and the map ξ given by

$$\xi(z) \;=\; \left\{\begin{array}{cc} e & \text{for } z\cdot e>0\\ 0 & \text{for } z\cdot e\le 0\end{array}\right\}.$$

Observe that ξ is the gradient of φ wherever the latter is differentiable. Obviously, there exists a sequence $\{\varphi_\nu\}_{\nu\uparrow\infty}$ in $C_0^\infty(I\!R^2)$ s. t.

$$\{(\varphi_\nu(z),\nabla\varphi_\nu(z))\}_{\nu\uparrow\infty} \quad \text{is bounded uniformly for bounded } z, \qquad (24)$$

$$(\varphi_\nu(z),\nabla\varphi_\nu(z)) \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; (\varphi(z),\xi(z)) \quad \text{for all } z. \qquad (25)$$

According to Lemma 3,

$$\Phi_\nu(z) \;=\; \varphi_\nu(z)\, z + \nabla\varphi_\nu(z)\cdot \mathrm{R}z\, \mathrm{R}z$$

is an entropy. (24) implies (22) and according to (25),

$$\begin{aligned}
\Phi_\nu(z) \;&\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \varphi(z)\, z + \xi(z)\cdot \mathrm{R}z\, \mathrm{R}z \\
&= \left\{ \begin{array}{cc} z\cdot e\, z + e\cdot \mathrm{R}z\, \mathrm{R}z & \text{for } z\cdot e > 0 \\ 0 & \text{for } z\cdot e \le 0 \end{array} \right\} \\
&= \left\{ \begin{array}{cc} |z|^2\, e & \text{for } z\cdot e > 0 \\ 0 & \text{for } z\cdot e \le 0 \end{array} \right\},
\end{aligned}$$

which turns into (23).

Lemma 5 *Let μ be a probability measure on $I\!R^2$ supported on S^1. Suppose it has the property*

$$\int \Phi\cdot \mathrm{R}\tilde\Phi\, d\mu \;=\; \int \Phi\, d\mu \cdot \int \mathrm{R}\tilde\Phi\, d\mu \quad \textit{for all entropies}\ \Phi, \tilde\Phi.$$

Then μ is a Dirac measure.

Proof of Lemma 5. According to Lemma 4, we are allowed to use the generalized entropies of the form (21). As μ is supported on S^1, this yields

$$e\cdot \mathrm{R}\tilde e\, \mu(\{z\cdot e > 0\}\cap\{z\cdot \tilde e > 0\}) \;=\; e\cdot \mathrm{R}\tilde e\, \mu(\{z\cdot e > 0\})\, \mu(\{z\cdot \tilde e > 0\})$$
$$\text{for all } e, \tilde e \in S^1$$

or

$$\mu(\{z\cdot e > 0\}\cap\{z\cdot \tilde e > 0\}) \;=\; \mu(\{z\cdot e > 0\})\, \mu(\{z\cdot \tilde e > 0\})$$
$$\text{for all } \tilde e \in S^1 - \{e, -e\} \text{ and all } e\in S^1.$$

Sending $\tilde e$ to e yields

$$\mu(\{z\cdot e > 0\}) \;\le\; \mu(\{z\cdot e > 0\})\, \mu(\{z\cdot e \ge 0\}) \quad \text{for all } e\in S^1$$

or

$$(\ \mu(\{z\cdot e > 0\}) \;=\; 0 \quad \text{or} \quad \mu(\{z\cdot e \ge 0\}) \;\ge\; 1\ ) \quad \text{for all } e\in S^1.$$

As μ is a probability measure, this implies

$$(\ \operatorname{supp}\mu \subset \{z\cdot e \le 0\} \quad \text{or} \quad \operatorname{supp}\mu \subset \{z\cdot e \ge 0\}\ ) \quad \text{for all } e\in S^1.$$

As the measure μ is concentrated on S^1, this forces it to be concentrated on a single point on S^1.

3 Compensated compactness and Young measures

Proof of the Proposition. We may focus on Proposition 2, since as explained in Section 1 it implies Proposition 1.
The first step is to show that for any entropy $\Phi \in C_0^\infty(I\!R^2)^2$,

$$\{\nabla\cdot[\Phi(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{is compact in } H^{-1}(\Omega). \tag{26}$$

According to Lemma 1, Lemma 2 and (4), there exists a $\Psi \in C_0^\infty(I\!R^2)^2$ such that

$$\nabla\cdot[\Phi(m_\nu)] \;=\; \Psi(m_\nu)\cdot\nabla(1-|m_\nu|^2) \quad \text{a. e. in } \Omega. \tag{27}$$

Since Ψ is bounded and according to (5), $\{(1-|m_\nu|^2)\,\Psi(m_\nu)\}_{\nu\uparrow\infty}$ converges to zero in $L^2(\Omega)$. As a consequence, $\{\nabla\cdot[(1-|m_\nu|^2)\,\Psi(m_\nu)]\}_{\nu\uparrow\infty}$ converges to zero in $H^{-1}(\Omega)$. Therefore, (26) would follow from

$$\{\nabla\cdot[\Phi(m_\nu)-(1-|m_\nu|^2)\,\Psi(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{is compact in } H^{-1}(\Omega), \tag{28}$$

which we show now: Thanks to (27), we have

$$\nabla\cdot[\Phi(m_\nu)-(1-|m_\nu|^2)\,\Psi(m_\nu)] \;=\; \nabla\cdot[\Psi(m_\nu)]\,(1-|m_\nu|^2) \quad \text{a. e. in } \Omega. \tag{29}$$

We observe that since Φ and Ψ are bounded and according to (5),

$$\{\Phi(m_\nu)-(1-|m_\nu|^2)\,\Psi(m_\nu)\}_{\nu\uparrow\infty} \quad \text{is uniformly integrable.} \tag{30}$$

Since $\mathrm{D}\Psi$ is bounded and according to (6),

$$\{\nabla\cdot[\Psi(m_\nu)]\,(1-|m_\nu|^2)\}_{\nu\uparrow\infty} \quad \text{is bounded in } L^1(\Omega). \tag{31}$$

A lemma by Murat [16] now states that thanks to (30) and (31), the identity (29) implies (28). This establishes the proof of (26).

In the second step, we apply the tools of Young measures and compensated compactness in the spirit of Tartar [18]. According to L. C. Young's theory of generalized functions (also called Young measures), there exists a non negative Borel measure μ_x such that for a subsequence

$$\int_\Omega\int \zeta(z,x)\,d\mu_x(z)\,dx \;=\; \lim_{\nu\uparrow\infty}\int_\Omega \zeta(m_\nu(x),x)\,dx \tag{32}$$

$$\text{for all } \zeta \in C_0^\infty(I\!R^2 \times \Omega),$$

with the understanding that the function $\Omega \ni x \mapsto \int \zeta(z,x)\, d\mu_x(z)$ is integrable for any $\zeta \in C_0^\infty(I\!R^2 \times \Omega)$ (see [18], [4], [5]). The family $\{\mu_x\}_{x\in\Omega}$ is called the Young measure associated to the subsequence $\{m_\nu\}_{\nu\uparrow\infty}$. According to (5), $\{|m_\nu|^2\}_{\nu\uparrow\infty}$ is uniformly integrable. Therefore, (32) can be improved to

$$\int_\Omega \int \zeta(z,x)\, d\mu_x(z)\, dx \;=\; \lim_{\nu\uparrow\infty} \int_\Omega \zeta(m_\nu(x),x)\, dx \tag{33}$$
$$\text{for all } \zeta \in C^\infty(I\!R^2 \times I\!R^2) \text{ with } \sup_{(z,x)\in \mathbf{R}^2\times\mathbf{R}^2} \frac{|\zeta(z,x)|}{1+|z|^2} < \infty.$$

By choosing $\zeta = \zeta(x)$ in (33), we see that

$$\int d\mu_x \;=\; 1 \quad \text{for a. e. } x \in \Omega. \tag{34}$$

Another generalization of (32) which comes for free is

$$\int_\Omega \int \zeta(z,x)\, d\mu(z)\, dx \;\le\; \limsup_{\nu\uparrow\infty} \int_\Omega \zeta(m_\nu(x),x)\, dx \tag{35}$$
$$\text{for all non negative } \zeta \in C^\infty(I\!R^2 \times I\!R^2).$$

By choosing $\zeta(z) = (1-|z|^2)^2$ in (35), we see that (5) implies

$$\text{supp}\mu_x \;\subset\; S^1 \quad \text{for a. e. } x \in \Omega. \tag{36}$$

Let $\Phi, \tilde\Phi$ be two entropies. According to our first step,

$$\{\nabla\cdot[\Phi(m_\nu)], \nabla\times[\mathrm{R}\tilde\Phi(m_\nu)] = \nabla\cdot[\tilde\Phi(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{are compact in } H^{-1}(\Omega).$$

Therefore by the div–curl Lemma of Murat and Tartar ([15] and [18]) the weak–* limit of the product of $\Phi(m_\nu)$ and $\mathrm{R}\tilde\Phi(m_\nu)$ in $L^1(\Omega)$ is the product of the weak limits in $L^2(\Omega)$ of $\Phi(m_\nu)$ resp. $\mathrm{R}\tilde\Phi(m_\nu)$. According to (32), these weak limits can be expressed in terms of the Young measure $\{\mu_x\}_{x\in\Omega}$; hence on the level of the Young measure, we obtain the commutation relation

$$\int \Phi\cdot \mathrm{R}\tilde\Phi\, d\mu_x \;=\; \left(\int \Phi\, d\mu_x\right)\cdot\left(\int \mathrm{R}\tilde\Phi\, d\mu_x\right) \quad \text{for a. e. } x \in \Omega.$$

Because of this and (34), (36), we may apply Lemma 5 to the effect of

$$\mu_x \text{ is a Dirac measure for a. e. } x \in \Omega.$$

This entails

$$\int |z|^2 \, d\mu_x(z) \;=\; |m(x)|^2 \quad \text{where} \quad m(x) \;=\; \int z \, d\mu_x(z) \quad \text{for all } x \in \Omega, \tag{37}$$

where according to (33), m is the weak-* limit of $\{m_\nu\}_{\nu\uparrow\infty}$ in $L^1(\Omega)$. As a consequence of (5), $\{|m_\nu|^2\}_{\nu\uparrow\infty}$ is uniformly integrable, so that m is the weak limit of $\{m_\nu\}_{\nu\uparrow\infty}$ in $L^2(\Omega)$. According to (37) and (33) for $\zeta(z,x) = |z|^2$, we have

$$\|m\|_{L^2(\Omega)} \;=\; \lim_{\nu\uparrow\infty} \|m_\nu\|_{L^2(\Omega)}.$$

As it is well–known, convergence of the norm strengthens weak convergence to strong convergence in $L^2(\Omega)$, so that

$$\lim_{\nu\uparrow\infty} \|m_\nu - m\|_{L^2(\Omega)} \;=\; 0.$$

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